Joining $k-$ and $l-$ recognizable sets of natural numbers

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Summary. We show that the first order theory of $\langle \mathbb{N}, +, V_k, V_l \rangle$, where $V_r : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ is the function which sends $x$ to $V_r(x)$, the greatest power of $r$ which divides $x$ and $k$, $l$ are multiplicatively independent (i.e. they have no common power) is undecidable. Actually we prove that multiplication is definable in $\langle \mathbb{N}, +, V_k, V_l \rangle$. This shows that the theorem of Büchi cannot be generalized to a class containing all $k-$ and all $l$-recognizable sets.

Introduction. As J.R. Büchi showed (see section 3.), a subset of $\mathbb{N}^n$ represented in base $k$ is recognizable on the alphabet $\{0, 1, \ldots, k-1\}^n$ if and only if it is definable in the first-order theory of $\langle \mathbb{N}, +, V_k \rangle$, where $V_k(x)$ is the greatest power of $k$ which divides $x$. This shows that the class of $k$-recognizable subsets of $\mathbb{N}^n$ ($n \in \mathbb{N}$) is closed under intersection, complementation, and projection. Hence a set is in the smallest class containing all $k$-recognizable sets and closed under intersection, complementation and projection if and only if it is definable in $\langle \mathbb{N}, +, V_k \rangle$.

A. Joyal asked to which extent it could be possible to generalize the above result joining $k$- and $l$-automata. I proved that if one takes the smallest class closed under intersection, complementation and projection which contains all $k$- and all $l$-recognizable subsets of $\mathbb{N}^n$ ($n \in \mathbb{N}$) (hence the definable subsets of $\langle \mathbb{N}, +, V_k, V_l \rangle$), then it contains multiplication. Therefore there is no machine specializing Turing machines by which exactly the sets in this class are recognized. Hence one cannot hope to generalize Büchi's theorem in this way.

In the first three sections we give definitions and results about automata, recognition and logic. In section 4. we reduce the main theorem to some technical result, which we prove in the last section.

1. Automata. Let $\Sigma$ be an alphabet, i.e. a finite set. $\Sigma^*$ will denote the set of words of finite length on $\Sigma$ containing the empty word $\lambda$ formed of no symbol. Any subset $L$ of $\Sigma$ will be called a language on the alphabet $\Sigma$.

Definition. Let $\Sigma$ be an alphabet. A $\Sigma$-automata $A$ is a quadruplet $(Q, q_0, \Gamma, T)$ where

$Q$ is a finite set, called the set of states,
The transition function $T$ can be extended to a function $T^* : Q \times \Sigma^* \to Q$ in the following way:

$$T^*(q, \sigma) = T(q, \sigma) \quad \text{for} \quad \sigma \in \Sigma$$
$$T^*(q, \alpha \sigma) = T(T^*(q, \alpha), \sigma) \quad \text{for} \quad \alpha \in \Sigma^* \quad \text{and} \quad \sigma \in \Sigma$$

Furthermore we have the following definitions.

**DEFINITION.** A word $\alpha \in \Sigma^*$ is said to be accepted by the $\Sigma$-automata $(Q, q_0, \Gamma, T)$ if $T^*(q_0, \alpha) \in \Gamma$.

**DEFINITION.** A language $L$ on $\Sigma$ is said to be $\Sigma$-recognizable if there exists a $\Sigma$-automata such that the set of words accepted by this automata is exactly $L$.

2. **Recognition over $\mathbb{N}$.** Let $\Sigma_k$ be the alphabet $\{0, 1, \ldots, k - 1\}$. For $n \in \mathbb{N}$ let $[n]_k$ be the word on $\Sigma_k$ which is the inverse representation of $n$ in base $k$, i.e. if $n = \sum_{i=0}^{s} \lambda_i k^i$ with $\lambda_i \in \{0, \ldots, k - 1\}$, then $[n]_k = \lambda_0 \cdots \lambda_s$.

It is also possible to represent tuples of natural numbers by words on $\Sigma_k^n$ in the following way. Let $(m_1, \ldots, m_n) \in \mathbb{N}^n$. Add on the right of each $[m_i]_k$ the minimal number of 0 in order to make them all of the same length and call these words $\omega_i$. Let $\omega_i = \lambda_{i1} \cdots \lambda_{is}$ where $\lambda_{ij} \in \Sigma_k$. We represent $(m_1, \ldots, m_n)$ by the word $(\lambda_{11}, \lambda_{21}, \ldots, \lambda_{1n}) (\lambda_{12}, \lambda_{22}, \ldots, \lambda_{2n}) \cdots (\lambda_{1s}, \lambda_{2s}, \ldots, \lambda_{ns}) \in (\Sigma_k^n)^*.$

**DEFINITION.** We say that a set $X \subseteq \mathbb{N}^n$ is $k$-recognizable if it is $\Sigma_k^n$-recognizable.

3. **Büchi’s Theorem.** Let $P_k(x)$ be the predicate (i.e. subset) on $\mathbb{N}$ defined by “$x$ is a power of $k$”. Let also as we said before $V_k : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ be the function which sends $x$ to $V_k(x)$, the greatest power of $k$ which divides $x$.

In [2, Theorem 9] Büchi states that a subset of $\mathbb{N}^n$ is $k$-recognizable if and only if it is definable in the first-order structure $< \mathbb{N}, +, P_k >$, i.e. defined by formulas built up from $=, +, P_k$ using $\wedge$ ("and"), $\neg$ ("not"), $\exists$ ("there exists a natural number such that ..."). Unfortunately, as remarked by McNaughton in [7], the proof is incorrect. Furthermore the statement has been disproved by Semenov in [11, Corollary 4]. Thanks to the work of Bruyère [1], we know that the ideas of Büchi can be used to show the following theorem. (See [1] for a proof among the lines of Büchi’s, [8] for a different proof or also [14]).

**THEOREM 3.1. Büchi’s Theorem** A set $X \subseteq \mathbb{N}^n$ is $k$-recognizable if and only if it is definable in the first order structure $< \mathbb{N}, +, V_k >$.

There is another version of Büchi’s Theorem in terms of weak monadic logic. Before we speak of it, let us give a useful definition and lemma.