Mathematical Progress as Synthesis of Intuition and Calculus

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In the following text the terms calculus and intuition are taken in a very broad sense. They are used as translations for the german words Kalkül and Anschauung.

1. The traditional conception of mathematics knows two sources of knowledge: intuition and calculus, or, otherwise taken, the senses - in particular the eyes - and the reason. Intuition leads to crude ideas, which can be true or false. The calculus on the other hand provides clearness and certainty: That, what is verified by calculation, is true (or demonstrated). Crude ideas (like the idea of a surface) are transformed into precise notions (like the notion of a differentiable manifold), which are linked by logical inferences. This process has been called exactification or (Carnap’s term) explication of a notion (Begriffserklärung).

Fig. 1.

Intuition is part of the art of inventing (ars inveniendi), calculus belongs to the art of justification (ars judicandi). The doctrine, which is generally accepted in the historiography of mathematics says, that this strict separation between intuition and calculus
was an achievement of the 19th century. It was the core of the new mathematical rigour; in the context of analysis it was obtained by the so-called arithmetization (F. Klein).

I want to treat here the relation between intuition and calculus in analyzing three examples taken from the history of mathematics in the 19th century. My thesis is:

Mathematics in its full historical content cannot be understood by such a reductionist program. Mathematical progress is always a synthesis out of intuition and calculus: Calculus without intuition is empty, that is without content, intuition without calculus is blind. (cf. the Critic of Pure Reason by I. Kant)

It is difficult to choose examples; the choice taken can always be criticized. I hope that I have found typical examples, which are interesting.

2. In the year 1799 a very young German mathematician published A new demonstration of the theorem that it is always possible to factorize every rational and entire function of one variable into factors, which are real and of first or second degree (Gauss, 1799). In this work Carl-Friedrich Gauss gave a nearly complete demonstration of the fundamental theorem of algebra. In using polar coordinates – Gauss doesn’t explain their geometrical signification (he is purely analytical) – the problem of the existence of a zero of the equation (or of the polynomials function) is reduced carefully to the existence of a point of intersection of two kinds of curves:

![Fig. 2.](image)

But the existence of this point is derived by a simple intuitional consideration which is illustrated by the picture Fig. 3.

This fact – that is the existence of the point of intersection – is treated by Gauss as something absolutely clear; he says nothing about it (whereas his paper is otherwise very explicit). Sixteen years later Gauss wrote in the preface to his second demonstration of the fundamental theorem (Gauss, 1815): My first demonstration depended partly on geometrical considerations... Here the term geometrical is synonymous with intuitional; it is opposed to analytical. Since all mathematics should be treated analytically, geometry