Equations for if-then-else

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In a pioneering paper [McCarthy 1963] John McCarthy described mathematical systems appropriate for the study of programming languages. The seventh section of that paper gives a universal-algebraic description of the ternary if-then-else operation \( \text{if}_p(f,g) \) which we now describe (with minor changes). There are two sorts, propositions \( p, q, r, \ldots \) which appear as the subscripted first variable and expressions \( f, g, h, \ldots \) which appear as the second two variables. There are also some nullary operations, the propositions 0, 1, 2 (for “false”, “true”, and “diverge”), as well as a proposition-valued ternary \( pqr \) (“if \( p \) is true then \( q \) else if \( p \) is false then \( q \) else diverge”). The eight equations, (labeled (2), (3), (5),..., (10) in McCarthy’s paper) are

\[
\begin{align*}
\text{if}_1(f,g) & = f \\
\text{if}_0(f,g) & = g \\
\text{if}_p(\text{if}_p(f,g), h) & = \text{if}_p(f,h) = \text{if}_p(f, \text{if}_p(g,h)) \\
\text{if}_{pqr}(f,g) & = \text{if}_p(\text{if}_q(f,g), \text{if}_r(f,g)) \\
\text{if}_p(\text{if}_q(f,a), \text{if}_q(b,g)) & = \text{if}_q(\text{if}_p(f,b), \text{if}_p(a,g)) \\
\text{if}_p(\text{if}_q(f,g), h) & = \text{if}_p(\text{if}_q(\text{if}_p(f,f), \text{if}_p(g,g)), h) \\
\text{if}_p(f, \text{if}_q(g,h)) & = \text{if}_p(\text{if}_q(f,g), \text{if}_p(h,h))
\end{align*}
\]

The standard semantics (from the point of view of programming languages) is provided by the following which we shall call \textit{McCarthy algebras}. Choose sets \( X, Y \). Model the propositions by \( \mathbb{T} \)-valued functions on \( X \), where \( \mathbb{T} = \{0,1,2\} \).
(0 = \textit{false}, 1 = \textit{true}, 2 = \textit{diverge}) is the set of truth values. Model expressions as partial functions from $X$ to $Y$. Let 0, 1, 2 be the corresponding constant functions, and let $\emptyset$ be the partial function which is everywhere undefined. Define

\[(pqr)(x) = \begin{cases} r(x) & \text{if } p(x) = 0; \\ q(x) & \text{if } p(x) = 1; \\ 2 & \text{if } p(x) = 2 \end{cases}\]

and define

\[(\text{if}_p(f, g))(x) = \begin{cases} g(x) & \text{if } p(x) = 0; \\ f(x) & \text{if } p(x) = 1; \\ \emptyset & \text{if } p(x) = 2 \end{cases}\]

(where we mean that $\text{if}_p(f, g)$ will be undefined e.g. if $p(x) = 0$ and $g(x)$ is undefined). Though it ultimately has no effect on the equational theory, this standard model can be described more generally in semilattice-assertional categories; see [Manes 1992a, Section 14].

It is routine to check that all McCarthy algebras satisfy his equations. McCarthy gave an algorithm which systematically uses these equations to reduce a term to a canonical form thereby solving the word problem for the generated variety. In fact, this variety is just that generated by the McCarthy algebras since it is easy to see that any two distinct normal forms have different semantics in one of these algebras.

Others have explored universal-algebraic models. See [Bloom and Tindell 1985], [Guesararian and Meseguer 1987] and [Mekler and Nelson 1987] as well as their bibliographies; but the standard models there are not those of McCarthy’s paper. Also note that [Bergman 1991] considers Boolean actions on sets which is the reduct of our 2-valued theory if $U$ and $\emptyset$ are deleted.