AN IMPROVED WAVE MODEL FOR MULTIDIMENSIONAL UPWINDING OF THE EULER EQUATIONS

P.L. Roe and Lisa Beard

Department of Aerospace Engineering
University of Michigan

INTRODUCTION

An essential element in the development of the truly multidimensional upwind schemes reviewed by Bram van Leer in these proceedings (see also [1]) is the reduction, in each computational cell, of the Euler equations to a set of independent scalar problems. In one dimension the difference of two successive nodal states \( u_{j+1} - u_j \) can be projected onto the eigenvectors of the local Jacobian matrix to produce a unique decomposition into simple waves, whose effects can be treated separately in the numerics. In \( n \) dimensions, \( n \) independent differences along the edges of a simplicial element define a linear variation of the fluid state within that element. However, the decomposition into simple waves is not now unique because such waves can have infinitely many orientations. The contribution made in this paper is to view the problem as one of kinematic analysis. This leads to a particularly natural decomposition.

ANALYSIS

Consider a region of space, filled with fluid, and small enough that all fluid properties can be taken to vary linearly within it. The gradients of the primitive variables can be assembled into a \( 5 \times 3 \) matrix, thus;

\[
P_x = \begin{bmatrix}
\rho_x & \rho_y & \rho_z \\
u_x & u_y & u_z \\
v_x & v_y & v_z \\
w_x & w_y & w_z \\
x & x & p_z
\end{bmatrix}.
\]

A “wave model” for this region will be defined as a set of plane waves, each satisfying the Euler equations, whose superposition will reproduce all the elements of (1).

The central block of nine terms in (1),

\[
D = \begin{bmatrix}
u_x & u_y & u_z \\
v_x & v_y & v_z \\
w_x & w_y & w_z
\end{bmatrix}.
\]

is known as the ‘deformation tensor’ [2]. A geometric interpretation of this tensor is that a set of fluid particles initially lying on the surface of a sphere with radius \( dr \) will, after a short time \( dt \), be found on the surface of an ellipsoid, whose equation is

\[
x^T[I - (D + DT)dt]x = (dr)^2.
\]
This ellipsoid depends only on the symmetric part $S = \frac{1}{2}(D + D^T)$ of $D$. The anti-symmetric part $A = \frac{1}{2}(D - D^T)$ of $D$, represents a rotation, and to first order does not affect the shape or the orientation of the ellipsoid. It is usual to conduct kinematic analyses in the (orthogonal) coordinates defined by the principal axes of the ellipsoid. These axes can be found by noting that they are the eigenvectors of the matrix $S$. In these axes the deformation tensor has a symmetric part that is purely diagonal, i.e.

$$D = S + A = \begin{bmatrix} u_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & w_z \end{bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}.$$ \hfill (4)

where the $\{\Omega_i\}$ are components of vorticity,

$$\Omega_x = \frac{1}{2}(w_y - w_z), \quad \Omega_y = \frac{1}{2}(u_z - u_x), \quad \Omega_z = \frac{1}{2}(v_x - u_y).$$

Note that the velocities are now also measured in the principal axis system. The axes can always be labelled to form a right-handed system such that $u_x \geq v_y \geq w_z$.

Each simple wave that might be proposed as an element of the wave model has an associated deformation tensor. We will only include waves having a deformation tensor of the correct form; those having no off-diagonal terms in their symmetric part. For example, any acoustic wave produces a deformation tensor that is purely symmetric, but the only cases without off-diagonal terms are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ \hfill (5)

which represent waves propagating along one of the principal axes.

The deformation tensor for a shear wave is always trace-free. It can be shown that the only cases with symmetric parts that are diagonal are

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$ \hfill (6)