On Paths in Networks with Valves

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Abstract: We consider networks $\mathcal{G}$ with the property that valves are installed in some of the nodes $v$. Entering $v$ by an arc $(u, v)$ and leaving it by another arc $(v, w)$ is only possible if $(u, v)$ and $(v, w)$ are connected by a valve adjustment. A path $[x_1, \ldots, x_k]$ is admissible if a valve adjustment exists which connects each arc $(x_k, x_{k+1})$ with $(x_{k+1}, x_{k+2})$.

We investigate the complexity of deciding whether there exists an admissible path from a node $s$ to another node $t$ of $\mathcal{G}$.

1. Introduction

This paper is about digraphs $\mathcal{G}$ which represent a network of pipes with valves; we investigate the complexity of deciding the existence of a path from a node $s$ to another node $t$ in $\mathcal{G}$. A heuristic solution of a similar problem was presented by U. Braun in [1]. He developed a program system for the numeric control of the "SIMATIC S5". This device is produced by the Siemens AG and automatically adjusts valves in pipe systems of breweries and refineries according to a route $P$ planned before.

U. Braun mainly considered directed graphs $\mathcal{G}$. They describe the situation that pumps are installed in every tube so that the stream of liquid can only run in one direction.

A further example for the great practical relevance of our decision problem is the situation of a computer network. Here the switches connecting particular pairs of ports can be interpreted as valves. In often occurs that the interconnections can only be used in one direction; therefore they should be represented by arcs in a digraph.

At the first sight, the desired path $P$ can be found with several well-known strategies like Depth-First-Search or Dijkstra's algorithm. But unlike the usual situation, not all arcs meeting at a node $v$ can be used arbitrarily if a valve is installed in $v$.

An example is Figure 1 where three possible valve adjustments of $v$ are shown; if $\vartheta_2$ is given then $v$ may only be entered via $r'_2$; after this, $v$ must be left via $r'_1$, $r'_3$ or $r'_4$ while $r'_2$ is forbidden.

So the question about the existence of $P$ is very difficult because the possible positions of all valves in the network must be considered.

Here we present a thorough investigation about the complexity of this decision. If $\mathcal{G}$ is a digraph the problem can be solved in polynomial time as long as all valves belong to a particular class $T_0$; otherwise the problem is NP-complete.
2. Notations and Definitions

Definition 2.1. The set of all natural numbers is \( \mathbb{N} := \{1, 2, 3, \ldots \} \); we write \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

If \( X, Y \) are two arbitrary sets then \( X \times Y \) is their cartesian product. Moreover, \( |X| \) is the cardinality of \( X \). At last, the set \( 2^X \) consists of all subsets of \( X \).

Definition 2.2. A digraph (without parallels and loops) is a pair \( G = (\mathcal{V}, \mathcal{R}) \) of sets where \( \mathcal{V} \) is the set of nodes and \( \mathcal{R} \subseteq (\mathcal{V} \times \mathcal{V}) \setminus \{(v, v) | v \in \mathcal{V}\} \) is the set of arcs.

For any arc \( r = (u, v) \in \mathcal{R} \) we write \( \alpha(r) := u \) and \( \omega(r) := v \). Moreover, \( u \) and \( v \) are called incident with \( r \).

If \( G = (\mathcal{V}, \mathcal{R}) \) is directed and \( v \in \mathcal{V} \) then we define the sets \( \mathcal{R}^{-}(v), \mathcal{R}^{+}(v) \subseteq \mathcal{R} \) as \( \mathcal{R}^{-}(v) := \{(x, y) \in \mathcal{R} | y = v\} \) and \( \mathcal{R}^{+}(v) := \{(x, y) \in \mathcal{R} | x = v\} \).

The quantities \( g^{-}(v) := |\mathcal{R}^{-}(v)| \) and \( g^{+}(v) := |\mathcal{R}^{+}(v)| \) are the in-degree and the out-degree of \( v \), resp.

Moreover, \( S^{-}(\mathcal{V}) := \{v \in \mathcal{V} | g^{-}(v) = 0\} \) is the set of all sources, and \( S^{+}(\mathcal{V}) := \{v \in \mathcal{V} | g^{+}(v) = 0\} \) is the set of all sinks.

Remark 2.3. Throughout our paper we only consider finite graphs.

Definition 2.4. (Paths in Graphs)

Given a digraph \( G \).

A path in \( G \) is a sequence \( P = [x_0, \ldots, x_l] \) with \( (x_i, x_{i+1}) \in \mathcal{R} \) for all \( \lambda = 0, \ldots, l - 1 \). We write \( \alpha(P) := x_0 \) and \( \omega(P) := x_l \).

If \( x_0 = x_l \) then \( P \) is called a cycle. We say that \( P \) is elementary iff all nodes \( x_0, \ldots, x_{l-1} \) are pairwise distinct and \( x_l \notin \{x_1, \ldots, x_{l-1}\} \); in particular, elementary cycles with \( x_0 = x_l \) are possible.

If \( P = [v_0, \ldots, v_l] \) is given then any path \( Q = [v_0, \ldots, v_\lambda] \) with \( \lambda \leq l \) is called a prefix of \( P \) ("\( P \) \). If \( Q = [v_\lambda, v_{\lambda+1}, \ldots, v_l] \) with \( 0 \leq \lambda' \leq \lambda'' \) then \( Q \) is a subpath of \( P \); this is written as \( Q \subseteq P \). In particular, \( P \) is a prefix and a subpath of itself.

Given two paths \( Q \) and \( Q' \) with \( \alpha(Q') = \omega(Q) \). Then the concatenation \( P := Q + Q' \) is the path that first uses \( Q \) and then traverses \( Q' \). The operation \( ' + ' \) is also defined for arcs; e.g., the path \( P = Q + r + r' + Q' \) uses the path \( Q \), the arcs \( r, r' \) and the path \( Q' \) in this order.

The set of all paths in \( G \) is written as \( \mathcal{P}(G) \). Moreover, if \( v, w \in \mathcal{V} \) then \( \mathcal{P}(v) \) is the set of all paths starting from \( v \), and \( \mathcal{P}(v, w) \) contains all paths from \( v \) to \( w \); every element of \( \mathcal{P}(v, w) \) is called a \( v-w \)-path.

3. The Complexity of Valve Problems

We consider the following situation: A valve is installed in some nodes \( v \) of a digraph \( G \); each valve adjustment connects exactly one incoming arc \( r^{-} \in \mathcal{R}^{-}(v) \) with some outgoing arcs \( r^{+} \in \mathcal{R}^{+}(v) \). This corresponds to the requirement that a stream of liquid may enter a valve only by one incoming pipe.

Definition 3.1. A (directed) valve graph is a triple \( (G, \mathcal{V}, \gamma) \) with the following properties:

\[ G = (\mathcal{V}, \mathcal{R}) \] is a directed graph. The set \( \mathcal{V} \subseteq \mathcal{V} \setminus (S^{-}(G) \cup S^{+}(G)) \) contains all valve nodes of \( G \). The third component is the valve function \( \gamma \); for every arc \( (u, v) \) with \( v \in \mathcal{V} \), the set \( \gamma(u, v) \) consists of all arcs \( (v, w) \) which can be used after \( (u, v) \). E.g., \( \gamma(\mathcal{V}) = \{r'_2, r'_4\} \) in Fig. 1.