Topological Relations  
Between Regions in $\mathbb{R}^2$ and $\mathbb{Z}^2^*$

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Abstract. Users of geographic databases that integrate spatial data represented in vector and raster models, should not perceive the differences among the data models in which data are represented, nor should they be forced to apply different concepts depending on the model in which spatial data are represented. A crucial aspect of spatial query languages for such integrated systems is the need mechanisms to process queries about spatial relations in a consistent fashion. This paper compares topological relations between spatial objects represented in a continuous (vector) space of $\mathbb{R}^2$ and a discrete (raster) space of $\mathbb{Z}^2$. It applies the 9-intersection, a frequently used formalism for topological spatial relations between objects represented in a vector data model, to describe topological relations for bounded objects represented in a raster data model. We found that the set of all possible topological relations between regions in $\mathbb{R}^2$ is a subset of the topological relations that can be realized between two bounded, extended objects in $\mathbb{Z}^2$. At a theoretical level, the results contribute toward a better understanding of the differences in the topology of continuous and discrete space. The particular lesson learnt here is that topology in $\mathbb{R}^2$ is based on coincidence, whereas in $\mathbb{Z}^2$ it is based on coincidence and neighborhood. The relevant differences between the raster and the vector model are that an object's boundary in $\mathbb{Z}^2$ has an extent, while it has none in $\mathbb{R}^2$; and in the finite space of $\mathbb{Z}^2$ there are points between which one cannot insert another one, while in the infinite space of $\mathbb{R}^2$ between any two points there exists another one.

1 Introduction

Spatial relations are significant ingredients of query languages for geographic information systems (GISs), where they are used to describe constraints among spatial objects to be retrieved or updated. Relations among spatial objects are less well understood than the commonly used relations among integers or strings [18] and

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attempts to formalize them have been rare. Recently, progress has been made on selected topics such as cardinal directions between point objects [14, 20] and some combinations of relations such as topological and direction relations [2, 25, 28].

Topological relations have been studied extensively in the past and there exists a comprehensive formalism [7, 8]. This model has become popular in the GIS community and has been used extensively for a variety of purposes. Svensson and Zhexue [37] incorporated it into a spatial-analysis query language. De Hoop and van Oosterom [27] designed a topological query language around these operators. Herring [26] discussed possible extensions of the model to cover topological relations between lines in $\mathbb{R}^2$. Egenhofer and Al-Taha [12] used the model to investigate various aspects of temporal changes of topological relations. Pigot [32] and Hazelton et al. [23] showed how the model applies to 3-dimensional objects; Hadzilacos and Tryfona [22] and Clementini et al. [4] showed how the model for regions behaves when applied to model topological relations between regions, lines, and points; and Egenhofer and Herring [9] extended the principle of boundary and interior intersections to include intersections with the exterior as well as a generic model for topological relations involving $n$-dimensional objects embedded in higher-dimensional spaces. Parts of these extensions are now being tested with human subjects to identify how closely the formalism models human cognition [30]. The sound model also enabled a number of advanced theoretical studies such as the formal derivation of the composition table for this set of relations [6] and comprehensive reasoning systems to detect inconsistencies in topological descriptions [11, 35]. A derivative of the method has also been successfully implemented in a commercial GIS [26].

This paper extends the scope of this model by investigating its usefulness for modeling topological relations among bounded objects embedded in the discrete space $\mathbb{Z}^2$. Its goal is to answer such questions as, “Are the topological relations between bounded objects in $\mathbb{R}^2$ and $\mathbb{Z}^2$ the same?” and if they are different then, “What are their differences?” By basing the investigations of raster relations on the same model as vector relations, we expect to make formal comparisons in the model, rather than giving intuitive interpretations.

Such an approach is a significant contribution towards designing spatial information systems that integrate what is usually called a “vector” and “raster” representation of spatial objects [16, 21], which is a topic that has gotten considerable attention through the discussions of integrating remotely sensed data with GIS data in vector format [13]. Much of this discussion has been at the level of data structures. The actual problems faced as one attempts to merge a data model for continuous space with a model of discrete space, are deeper in nature. To date, there exist only a few approaches that try to combine the two views at a conceptual level [31]. In order to come up with an integrated model it is definitely necessary to formalize the differences at the conceptual level, rather than the level of particular data structures or implementations. This paper contributes towards such an integrated data model for geographic databases, as it identifies topological properties that are common or different between the two views. As such, the results of this paper will help in gaining a better understanding of the differences between raster and vector space.