Logical Definability of NP–Optimisation Problems with Monadic Auxiliary Predicates

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Abstract. Given a first–order formula $\varphi$ with predicate symbols $e_1 \ldots e_i, s_0, \ldots, s_r$, an NP–optimisation problem on $<e_1, \ldots, e_i>$–structures can be defined as follows:
for every $<e_1, \ldots, e_i>$–structure $G$, a sequence $<S_0, \ldots, S_r>$ of relations on $G$ is a feasible solution iff $<G, S_0, \ldots, S_r>$ satisfies $\varphi$, and the value of such a solution is defined to be $|S_0|$. In a strong sense, every polynomially bounded NP–optimisation problem has such a representation, however, it is shown here that this is no longer true if the predicates $s_1, \ldots, s_r$ are restricted to be monadic. The result is proved by an Ehrenfeucht–Fraïssé game and remains true in several more general situations.

1 Introduction

With his seminal paper [Fa74], Ronald Fagin introduced finite model theory into computational complexity theory. He provided a non–computational characterisation of NP by proving that, for any finite signature $\sigma$, a set $L$ of finite $\sigma$–structures is in NP iff there is an extension $\sigma'$ of $\sigma$ and a first–order sentence $\varphi$ over $\sigma'$ such that a $\sigma$–structure $G$ belongs to $L$ if and only if it can be extended to a $\sigma'$–structure which satisfies $\varphi$.

This result has since been developed further, mainly in two directions. On the one hand similar characterisations have been found for many other complexity classes (for a survey see [Im89]), on the other hand the possibility of restricting the syntactical form of the formula $\varphi$ has been used to differentiate between problems which cannot be distinguished by their computational complexity (cf.[dR87]).

One motivation for this line of research is the fact that in finite model theory there are means of showing separation results, something we do not seem to be able to do in computational complexity. However, so far, none of the model theoretic separations have carried over to a separation of complexity classes.

Recently, Papadimitriou and Yannakakis took this approach of looking at restricted formulae one step further, and applied it to NP–optimisation problems [PY91]. Their aim was to find an explanation for the different behaviour of such problems with respect to polynomial–time approximation algorithms. The main result of their paper is the approximability to within a constant factor of all those optimisation problems which are defined in a certain way by a formula of the form $\forall x \exists y \varphi$. Their way of defining optimisation problems by first–order formulae can be rephrased as follows.

$^1$ Here $\forall x$ is short for $\forall x_1, \ldots, \forall x_m$, for some $m \geq 0$, similarly, $\exists y$. 
Let $\sigma$ be a signature and $\sigma'$ an extension of $\sigma$, such that $\sigma' - \sigma = \langle s_0, s_1, \ldots, s_r \rangle$. Then every first order sentence $\varphi$ over $\sigma'$ gives rise to the following optimisation problem:

Given A $\sigma$–structure $G$.

Wanted A $\sigma'$–extension $\langle G, s_0, \ldots, s_r \rangle$ of $G$ which maximises (or minimises) $|s_0|$, subject to the condition that $\langle G, s_0, \ldots, s_r \rangle \models \varphi$.

As an example consider the problem MAXCLIQUE:

Given A graph $G = (U, E)$.

Wanted A maximum–size subset $V \subseteq U$ such that $G$ induces a complete graph on $V$.

If we take $\sigma$ to consist of one binary predicate symbol $e$, $\sigma' - \sigma$ of one unary symbol $v$, we can express the fact that the elements of $V$ form a clique by

$$\varphi \equiv \forall x \forall y (x \neq y \land v(x) \land v(y)) \rightarrow (e(x, y) \lor e(y, x)).$$

Consequently the clique number $\omega(G)$ can be expressed as:

$$\omega(G) = \text{max} \{|V| / \langle G, V \rangle \models \varphi\}.$$

Given this definition of optimisation problems by formulae, Papadimitriou and Yannakakis' result can be stated in the following way.

1.1 Theorem ([PY91])

Let $\sigma' - \sigma = \langle s_0, \ldots, s_r \rangle$, and let $\psi$ be a quantifier–free first–order formula over $\sigma' - \{s_0\}$ with free variables $\bar{x}, \bar{y}$. If $\varphi = \forall \bar{x} \exists \bar{y} s_0(\bar{x}) \rightarrow \psi$, then the maximisation problem defined by $\varphi$ can be solved in polynomial time approximatively to within a constant factor.

How strong is this result? Although the restriction for $\varphi$ seems rather severe compared to full first–order logic, the example above shows that we cannot hope for much better results along these lines. In fact the formula for the clique problem violates the restrictions of the theorem "only just"; however, there seems to be little hope of finding a polynomial–time approximation algorithm with guaranteed performance for MAXCLIQUE\(^2\). On the other hand, at first sight, it is not clear which class of optimisation problems can be defined at all by first–order formulae in the way described above. A first answer to this question was given by Kolaitis and Thakur in [KT90], where they showed that the optimal value of every NP–optimisation problem (with polynomially bounded values) equals the optimal value of an optimisation problem which is defined, in the way sketched above, by some first–order formula $\varphi$, in fact by a formula of the form $\forall \bar{x} \exists \bar{y} \psi$. Their result is, however, unsatisfactory in that the logically defined version of a given NP–optimisation problem might only have the same optimal

\(^2\) In fact, in a recent paper [ALMSS], Arora et al show that approximation to within $n^*$ would imply $P = NP$. 