On the Complexity of Graph Embeddings
(Extended Abstract)

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Abstract. It is known that embedding a graph $G$ into a surface of minimum genus $\gamma_{\text{min}}(G)$ is $NP$-hard, whereas embedding a graph $G$ into a surface of maximum genus $\gamma_{\text{M}}(G)$ can be done in polynomial time. However, the complexity of embedding a graph $G$ into a surface of genus between $\gamma_{\text{min}}(G)$ and $\gamma_{\text{M}}(G)$ is still unknown. In this paper, it is proved that for any function $f(n) = O(n^e)$, $0 \leq e < 1$, the problem of embedding a graph $G$ of $n$ vertices into a surface of genus at most $\gamma_{\text{min}}(G) + f(n)$ remains $NP$-hard, while there is a linear time algorithm that approximates the minimum genus embedding either within a constant ratio or within a difference $O(n)$. A polynomial time algorithm is also presented for embedding a graph $G$ into a surface of genus $\gamma_{\text{M}}(G) - 1$.

1 Introduction

Minimum genus $\gamma_{\text{min}}(G)$ of a graph $G$ is defined to be the smallest integer $k$, such that $G$ has a 2-cell embedding into an orientable surface of genus $k$. Maximum genus $\gamma_{\text{M}}(G)$ of a graph $G$ is defined to be the largest integer $k$ such that $G$ has a 2-cell embedding into an orientable surface of genus $k$.

Embedding a graph into topological surfaces is a fundamental, yet very difficult, problem. The computational complexity of constructing the embeddings of a graph into surfaces of different genus is not well-understood. Not much progress had been made until very recently. Algorithms have been developed for embedding a graph into the minimum genus surface as well as into the maximum genus surface. It was demonstrated by Furst, Gross, and McGeoch [5] that a maximum genus embedding of a graph can be constructed in polynomial time. The algorithm presented in [5] is based on a characterization of the maximum genus of a graph given by Xuong [14]. On the other hand, research shows that constructing minimum genus embeddings of a graph is more difficult. For the class $C_g$ of graphs whose minimum genus is bounded by a constant $g$, Filotti, Miller, and Reif [4] derived an $O(n^{O(g)})$ time algorithm for determining the minimum genus of a graph, which was improved recently by Djidjev and Reif [2] who developed an algorithm of time $O(2^{O(g)}n^{O(1)})$. The celebrated work of Robertson and Seymour [12] gives an $f(g)n^2$ time algorithm for determining the

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minimum genus of a graph in the class $C_g$, where $f(g)$ is a very fast growing function. A result for general graphs was obtained recently by Thomassen [13], who showed that the following problem is $NP$-complete: given a graph $G$ and an integer $k$, is $\gamma_{\text{min}}(G) \leq k$? Note that this question was one of the remaining basic open problems, listed by Garey and Johnson [6].

At the end of their paper [5], Furst, Gross, and McGeoch posed several open problems, one of them is asking the complexity of embedding a graph into a surface of genus $k$, where $\gamma_{\text{min}}(G) < k < \gamma_{\text{M}}(G)$.

In the present paper, we will provide a partial answer to the above question. Our main results are: 1) for any function $f(n) = O(n^\epsilon)$, where $0 \leq \epsilon < 1$ is a fixed constant, constructing an embedding of a graph $G$ of $n$ vertices into a surface of genus at most $\gamma_{\text{min}}(G) + f(n)$ is still $NP$-hard; 2) a polynomial time algorithm for embedding a graph $G$ into a surface of genus $\gamma_{\text{M}}(G) - 1$; and 3) a linear time algorithm that, given a graph $G$ of $n$ vertices, constructs an embedding $\Pi(G)$ of $G$ such that either the genus of $\Pi(G)$ is less than $\gamma_{\text{min}}(G) + O(n)$ or the ratio between the genus of $\Pi(G)$ and the minimum genus $\gamma_{\text{min}}(G)$ is bounded by a constant.

We point out that our first two results are not simple consequences of the process of locally altering an embedding of a graph and decreasing the embedding genus. In fact, the process of decreasing embedding genus by locally altering the embedding is not always possible. As demonstrated by Gross and Rieper [9], there are non-minimum genus embeddings of some graphs, on which no local alteration will decrease the embedding genus. It is in fact unknown whether there exist a region $R$ between the minimum genus and the maximum genus and a polynomial time algorithm $A_R$ such that given an embedding of a graph $G$ into a surface of genus $k$, where $k$ is within the region $R$, the algorithm $A_R$ constructs a genus $k - 1$ embedding of the graph $G$.

Our first result of the intractability of embedding a graph $G$ into a surface of genus at most $\gamma_{\text{min}}(G) + f(n)$, where $f(n) = O(n^\epsilon)$, $0 \leq \epsilon < 1$, is based on a simple graph operation, the bar amalgamation. The contribution of the second result is to demonstrate that given a graph $G$, there exists a special maximum genus embedding of $G$, which can be constructed in polynomial time, such that it is always possible to decrease the embedding genus by locally altering the embedding. Our third result is based on a simple combinatorial analysis, and it is in contrast to the first result in some sense: the first result demonstrates that it is hard to approximate the minimum genus of a graph within a difference $O(n^\epsilon)$, $0 \leq \epsilon < 1$, while the third result claims that we can always approximate the minimum genus of a graph easily either within a constant ratio or within a difference $O(n)$.

The paper is organized as follows. In Section 2, we introduce the necessary background on theory of graph embeddings. The intractability of embedding a graph $G$ into a surface of genus at most $\gamma_{\text{min}}(G) + f(n)$, where $f(n) = O(n^\epsilon)$, $0 \leq \epsilon < 1$, is demonstrated in Section 3. A polynomial time algorithm for constructing an embedding of genus $\gamma_{\text{M}}(G) - 1$ for a graph $G$ is presented in Section 4. The approximability of graph minimum genus is discussed in Section 5.