On Fat Partitioning, Fat Covering and the Union Size of Polygons*
(extended abstract)
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Abstract

The complexity of the contour of the union of simple polygons can be $O(n^2)$ in general. In this paper, a necessary and sufficient condition is given for simple polygons which guarantees smaller union complexity. A $\delta$-corridor in a polygon is a passage between two edges with width/length ratio $\delta$. If a set of polygons with $n$ vertices in total has no $\delta$-corridors, then the union size is $O((n \log \log n)/\delta)$, which is close to optimal in the worst case. The result has many applications to basic problems in computational geometry, such as efficient hidden surface removal, motion planning, injection molding, etc. The result is based on a new method to partition a simple polygon $P$ with $n$ vertices into $O(n)$ convex quadrilaterals, without introducing angles smaller than $\pi/12$ radians or narrow corridors. Furthermore, a convex quadrilateral can be covered (but not partitioned) with $O(1/\delta)$ triangles without introducing small angles. The maximum overlap of the triangles at any point is two. The algorithms take $O(n \log^2 n)$ and $O(n \log^2 n + n/\delta)$ time for partitioning and covering, respectively.

1 Introduction

The primary motivation of this research is to determine for what sets of geometric objects (closed curves), the contour of the union has small complexity. When the union size is small, many geometric problems can be solved more efficiently and with simpler algorithms than in the general case. We give some examples later in this section.

Upper bounds on the union size have been found for several types of objects. Kedem et al.[6] show that the contour of the union of a set of $n$ pseudo-discs in the plane has linear description size (a set of pseudo-discs is a set of simply connected regions of which any two boundaries intersect at most twice). It is easy to see that the contour of the union of a set of $n$ isothetic rectangles can have $\Omega(n^2)$ connected components, and therefore quadratic description size, by placing them in a grid-like pattern. Since two isothetic rectangles intersect at

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most four times, the question arises what the maximum union size is of sets of unbounded regions of which every two boundaries intersect at most three times. This case was settled by Edelsbrunner et al.[4], who show that the contour size is $O(n\alpha(n))$, and there are $O(n\alpha(n))$ connected components in the contour (where $\alpha(n)$ is the extremely slowly growing functional inverse of Ackermann’s function). These bounds are tight in the worst case. Some other results on the maximum union size were obtained by Alt et al.[2].

Recently, Matoušek et al.[9] observed that for triangles, a quadratic lower bound example can only be constructed if the triangles have sharp angles. They proved that for a set of triangles of which any interior angle is at least $\delta$, for some constant $\delta > 0$, the union determines only $O(n)$ holes, and the contour size is $O(n \log \log n)$. Notice that two such triangles can intersect six times. The constant of proportionality in these results is $O(1/\delta^3)$, whereas the best known lower bound example gives a constant $\Omega(1/\delta)$.

In this paper we extend the results from [9] to the case of simple polygons. Also, we prove that the dependency on $\delta$ is $\Theta(1/\delta)$ for fat triangles and polygons. For polygons, the fatness condition that each angle is bounded from below by a constant clearly is not good enough, because the lower bound example with rectangles still holds. To obtain a necessary and sufficient condition to bound the union size of simple polygons, we make the following definitions (see Figure 1):

**Definition 1** For any $0 < \delta \leq 1$, a $\delta$-corridor is a convex quadrilateral $Q$ with vertices $p_1, p_2, p_3, p_4$ such that $\angle p_1p_2p_3 = \angle p_2p_3p_4$ and $\angle p_3p_4p_1 = \angle p_4p_1p_2$, and $|p_1p_2| = |p_3p_4| = \delta \cdot \max\{|p_2p_3|, |p_1p_4|\}$.

For any $0 < \delta \leq 1$, a simple polygon $P$ (or any set of edges) is $\delta$-wide if for any two edges $e$ and $e'$ of $P$, and any four points $p_1, p_2 \in e$ and $p_3, p_4 \in e'$ that are the vertices of a $\gamma$-corridor $Q$ such that $\text{interior}(Q) \subseteq \text{interior}(P)$, it follows that $\gamma \geq \delta$.