Abstract: We establish that a set of graphs generated by a "Vertex replacement" graph grammar can be generated by a "Hyperedge replacement" one iff its graphs do not contain arbitrarily large complete bipartite graphs $K_{n,n}$ as subgraphs, iff its graphs have a number of edges that is linearly bounded in terms of the number of vertices. These properties are decidable by means of an appropriate extension of the theorem by Parikh that characterizes the commutative images of context-free languages.

Introduction

Among the many notions of graph grammars which have been defined, some of them can be called context-free because their derivation sequences can be adequately described by derivation trees, and because the sets they generate form the least solution of systems of equations associated with them. These ideas are developed in [3]. There are two main types of context-free graph grammars, the hyperedge replacement grammars (HR for short) that generate graphs or hypergraphs by using hypergraphs as righthand sides of production rules and sentential forms [1,19]), and the vertex replacement grammars (VR for short) based on NLC-rewriting [17,20]. The boundary NLC grammars [2, 21-23], the confluent NLC grammars [3], the confluent edNCE grammars [15] and the separated handle hypergraph grammars [12], which all generate sets of simple graphs, are VR grammars.

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The generative powers of these two types of grammars have been compared in several papers [2,12,16,22], and we improve some results of these papers. Let us recall some results:

**Theorem 1** [12,15]: Every HR set of simple graphs is VR.

The following fact gives a characterization of the sets of graphs that are VR but not HR.

**Theorem 2** [11]: A VR set of graphs is HR iff it is a subset of some HR set iff it does not contain some clique \( K_n \) as a minor, iff it does not contain some complete bipartite graph \( K_m,n \) as a minor.

A graph \( G \) is a minor of a graph \( H \) if it is obtained from a subgraph of \( H \) by edge contractions; saying that a set of graphs \( L \) does not contain \( G \) as a minor means that no \( H \) in \( L \) contains \( G \) as a minor. A set of graphs is sparse if its members have a number of edges that is linearly bounded in terms of the number of vertices.

**Main Theorem**: A VR set of graphs is HR iff it does not contain some complete bipartite graph \( K_{n,n} \) as a subgraph, iff it is sparse. This is decidable.

We now present the three tools used in the proofs. The first one is a way to "factorize" graphs. The inverse of the factorization is called the "expansion" and is akin to the removal of \( \varepsilon \)-transitions in finite state automata. We prove that a set of simple graphs is VR iff it is the set of expansions of the graphs of some HR set. Our second tool is an extension of Parikh's Theorem to context-free sets of graphs that uses monadic second-order logic: instead of just counting vertex or edge labels like in [19, Chap.IV], we count the cardinalities of sets defined by monadic second-order formulas. Let us recall briefly the rôle of logic. A graph \( G \) can be described by (logical) structures of two types denoted by \( |G|_1 \) and \( |G|_2 \) respectively. The first structure has a domain consisting of the set of vertices of \( G \), and relations expressing the existence of edges between vertices. The structure \( |G|_2 \) has a domain consisting of vertices and edges; its relations express the incidences. For every set of graphs \( L \), we let \( |L|_i := \{|G|_i / G \in L\} \), for \( i = 1,2 \). Roughly speaking, the structures \( |G|_1 \) are adequate for expressing properties of graphs of VR sets by formulas of monadic second-order logic (MS formulas for short, i.e., by first-order formulas augmented with quantifications on sets), whereas the structures \( |G|_2 \) are adequate for describing those of graphs of HR sets. As third tool, we shall use transformations from graphs to graphs that are specified by MS formulas, where graphs are represented as explained above by relational structures. Such mappings will be called *definable graph transductions*. We shall handle VR and HR sets of graphs via the characterizations of [11,15] stating that a set \( L \) is...