1 Introduction

Exact solutions have a long and distinguished history within the development of General Relativity. Their study began with the exterior and interior Schwarzschild solutions which were discovered soon after Einstein published his field equations. After the derivation of the Weyl solutions a long hiatus set in as far as asymptotically flat solutions were concerned. There was, of course, considerable activity in the realm of algebraically special solutions or solutions admitting vectors with special properties. It was while looking for solutions with a particular structure of the metric that Kerr discovered his celebrated solution. About ten years later Tomimatsu and Sato found their series of exact axisymmetric stationary solutions.

This triggered a renewed interest in stationary axisymmetric solutions, a field which had been dormant because the equations were considered wellnigh insolvable. The effort was helped by the development of solution generating techniques, Bäcklund transformations or the inverse scattering method, for other non-linear partial differential equations. Finally several groups around the world published their approaches to axisymmetric stationary vacuum solutions. For a survey the reader is referred to Ref. 1. All solution generating techniques are equivalent in the sense that almost all solutions can be generated. It should be noted, that the techniques can be applied to any situation with two commuting Killing vectors, in particular also to colliding plane gravitational waves, cf. Ref.2.

On the other hand, we who are working in the realm of exact solutions should keep in mind that writing down a metric which solves Einstein’s equations is only the first part of the task. To paraphrase Kinnersley [3], we should not leave our newborn metric wobbling on its Vierbein without any visible means of interpretation.

In this series of lectures we give an overview of the HKX transformations and relate them to what, in a more general setting, are called linear problems.
As an example of a solution we describe the double Kerr solution. This is one of the solutions which demonstrate that relativistic angular momentum interaction can balance two masses against their gravitational attraction.

In addition to the above we also outline the definition of multipole moments in General Relativity and show how the first few of them can be calculated by an expansion of the Ernst potential. Furthermore we describe some properties of metrics which can be expressed as rational functions of prolate spheroidal coordinates.

On a personal note, I should like to express my appreciation to the organizers of the meeting at El Escorial for providing such a splendid and congenial environment. The workshop was a most pleasant experience for which I wish to thank Dr. F. J. Chinea.

2 Generating Solutions by HKX Transformations

We are concerned with space-times admitting two commuting Killing vectors. One of them is assumed to be spacelike with closed orbits, the other one should be timelike. It follows that the metric can be written in the Papapetrou-Lewis form

\[ ds^2 = \frac{1}{f} \left[ e^{2\gamma}(dx^2 + dy^2) + \rho^2 d\phi^2 \right] - f(dt - \omega d\phi)^2 \]  

(2.1)

\( \varphi \) and \( t \) are the Killing coordinates and the functions \( f, \omega, \rho \) and \( \gamma \) depend only on the non-ignorable coordinates \( x \) and \( y \). We shall use the derivative operators

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \quad \tilde{\nabla} = \left( \frac{\partial}{\partial y}, -\frac{\partial}{\partial x} \right), \quad \partial = \partial x + i\partial y \]  

(2.2)

One of the Einstein equations reads

\[ \nabla^2 \rho = 0 \]  

(2.3)

This equation implies the existence of a function \( z \) defined up to an additive constant by

\[ \nabla \rho = \tilde{\nabla} z \]  

(2.4)

Because the gradients of \( \rho \) and \( z \) are orthogonal and of equal magnitude one can use \( \rho \) and \( z \) instead of \( x \) and \( y \) as coordinates without altering the form of the metric. \( \rho \) and \( z \), are known under the name of Weyl coordinates. It should be noted that the Weyl coordinates are uniquely determined by the geometry, \( \rho \) as being the volume element of the 2-surfaces swept out by the Killing vectors and \( z \) as the conjugate function via (2.4). Consequently any statement about metric functions formulated in Weyl coordinates is ipso facto invariant.

One of the other field equations reads

\[ \nabla \frac{f^2}{\rho} \nabla \omega = 0 \]