1 Introduction

Several ways of solving the vacuum Einstein equations with two Killing vectors, usually written as the Ernst plus other equations, were found around the year 1978. Most of these accomplished the task by solution generation methods, in which a new solution was found by a technique starting with a known solution. One of these techniques employed a Bäcklund transformation [2,3,5]. It would be very useful if one could find a similar Bäcklund transformation (BT) for the equations with one Killing vector. We describe here a search for such a BT; we note, however, that the existence of such a transformation may be judged unlikely because most such BT’s apply only to equations with two independent variables.

An established method for search for a BT is that of Wahlquist and Estabrook (WE) [6,7]. To see how that is used, we review it here as applied to the Ernst equation. However, we will consider here only the first half of the method, which involves searching for a “prolongation structure” (PS).

2 The Ernst Equation in Differential Forms

To set up the Ernst equation, we choose a metric in standard form:

$$ds^2 = -f(dt + wd\omega)^2 + p^{-2}f^{-2}(dp^2 + dz^2) .$$  \hspace{1cm} (1)

We define a potential $\phi$ by, where subscripts mean derivation:

$$\phi_\rho = \rho^{-1}f^2\omega_z, \quad \phi_z = -\rho^{-1}f^2\omega_\rho .$$  \hspace{1cm} (2)

Then the Ernst potential is $E = f + i\phi$ and the Ernst equation is:
$E_{\rho\rho} + \rho^{-1}E_{\rho} + E_{zz} = f^{-1}(E_{\rho}^2 + E_{\phi}^2)$ \hspace{1cm} (3)

It is convenient to write it in terms of differential forms. We define a linear Hodge star operator $\star$ by $\star d\rho = dz$ and $\star dz = -d\rho$; then the Ernst equation may be written as two 2-form equations:

$$d(\star df) + \rho^{-1}d\rho \wedge \star df = f^{-1}(df \wedge \star df - d\phi \wedge \star d\phi)$$ \hspace{1cm} (4)

$$d(\star d\phi) + \rho^{-1}d\phi \wedge \star d\phi = 2f^{-1}df \wedge \star d\phi$$ \hspace{1cm} (5)

These can be satisfied formally by taking $\omega$ as a potential and using a new potential $\eta$. We write (2) as

$$\star d\phi = \rho^{-1}f^2d\omega$$ \hspace{1cm} (6)

and also write

$$\star df = \rho^{-1}f(d\eta + \omega d\phi)$$ \hspace{1cm} (7)

$\eta$ and $\omega$ are not arbitrary, since they must satisfy their own equations. Using the various variables, one can now introduce a set of six 1-forms $\xi_k$. The exterior derivatives of these 1-forms, and the Ernst equation itself, can be written in terms of products of the $\xi_k$ with constant coefficients. The equations for the $\xi_k$ are denoted as an ideal $I$. We do not write them out here because they are given in the references [2,3,4].

### 3 Prolongation Procedure for the Ernst Equation

Application of the WE method now proceeds as follows. We write a column vector of 1-forms,

$$\Omega = -dq + (B_k \xi_k)q$$ \hspace{1cm} (8)

where $q$ is a column of 0-forms (functions), the six $B_k$ are square matrices, and the $\xi_k$ are the 1-forms introduced above. There is a sum on $k$. The dimension of the $q$ space is not specified at this point; it is determined by the size of the representation discussed below. One now requires that the exterior derivative of $\Omega$ is included in the augmented ideal made up of $I$ and $\Omega$ itself:

$$d\Omega \subset \{I, \Omega\}$$ \hspace{1cm} (9)

Substitution of $\Omega$ into this equation gives a set of equations involving the $\xi_k$ and $B_k$. The requirement that these be in the ideal $I$ can most easily be satisfied by substituting for the $d\xi_k$ and by replacing certain products of the $\xi_k$ by their equivalent from the equations in $I$. Alternatively, one requires that $d\Omega$ be written as linear combinations of $\Omega$ and the equations in $I$. The resultant equations for the matrices $B_k$ can be solved in terms of an incomplete Lie algebra. (Note: for this problem we must take the matrices to be functions of a variable $\zeta$, which is invariant under the invariance group of the Ernst equation.) If one constructs a representation of the incomplete Lie algebra, then setting $\Omega$ equal to zero in (8) provides what is called a PS for the Ernst equation.