Presymplectic Manifolds and Conservation Laws

V. Liern 1, J. Olivert 2

1Departamento de Economía Financiera y Matemática, Universitat de València, 46010-València, Spain.
2Departament de Física Teòrica, Universitat de València, 46100-Burjassot (València), Spain

Abstract: In this paper we make use of a new structure called seeded fibre bundle. This allows us to combine the symplectic formalism and general relativity. A theorem of existence is obtained and some examples and properties are studied.

1 Introduction

In order to interpret the interaction of particles from a mathematical point of view, and within a non-quantum formalism, it is associated to a type of principal fibre bundles provided with connection. However, a particle which describes a specific movement is mathematically associated to a symplectic manifold on which a dynamic group acts. This association is an formalism independent of the observation, or better said, independent from the modification produced by the interaction that acts on the particle. However, this formalism can hardly be extended to the particles on which a given interaction acts, because in a symplectic manifold it can hardly be given a connection in a canonical way, and even less one that modifies the characteristic foliation of the symplectic manifold.

With the aim to be able to relate these ideas we need a new structure (the seeded fibre bundle one) which was introduced in the Ref. [7]. In that paper we proved that the results of elementary particles given by Souriau in Ref. [10] can be extended to general relativity. Furthermore, we showed that the class of the new fibre bundles was not empty. However, we did not give any theorem of existence.

In this work we prove the existence of these fibre bundles and we obtain, as a consequence of the theorem of existence, that a dynamic group in the symplectic manifold (the fibres) is also dynamic in the family of presimplectic manifolds of the seeded structure. Furthermore, it provides us with a method of construction that is used in the example given.
Finally, we study an example from the group $SU(2, \mathbb{C})$, and we apply Noether's Theorem in order to obtain some conservation laws. One of them is described by the following function:

$$f(\theta) = 4\sin^2 \frac{\theta}{2},$$

which we call strangeness function$^1$.

2 Seeded Fibre Bundles: Definition, Existence and Properties

Definition 2.1 Let $\lambda = (P, M, \pi, G)$ be a principal $G$–bundle provided with connection, $(U, \sigma_U)$ a symplectic Hausdorff manifold left $G$–space. $\lambda[U] = (P_U, M, \pi_U, G)$, a fibre bundle with fibre type $U$ associated to $\lambda$, is called seeded fibre bundle$^2$ if for every $x \in M$ exists $(V_x, \sigma_x) \subset P_U$ presymplectic regular manifold which satisfies:

(a) $\exists \Psi_x : V_x \longrightarrow \pi_U^{-1}(x)$ surjective submersion verifying

$$\pi_U^{-1}(x) = \frac{V_x}{\ker \Psi_x}.$$  

(b) Given $x, y \in M$, if $V_x \cap V_y \neq \emptyset$, then $V_x = V_y$.

(c) $\ker \sigma_{xw} \subset Q_w$, $Q$ being the horizontal distribution and $w \in V_x$.

Theorem 2.2 Let $\lambda = (P, M, \pi, G)$ be a principal $G$–bundle provided with connection, $(U, \sigma_U)$ a symplectic Hausdorff manifold. The fibre bundle associated to $\lambda$ with fibre type $U$ is a seeded fibre bundle if and only if a foliation $S$ contained in the horizontal distribution exists.

Proof. Let $\lambda[U] = (P_U, M, \pi_U, G)$ a seeded fibre bundle provided with connection (the induced by $\lambda$) and presymplectic manifolds family $\{(V_x, \sigma_x)\}_{x \in M}$. Then $E = \bigcup_{x \in M} V_x$ is a submanifold of $P_U$, as it is a disjoint union of submanifolds. Let us see that it is also presymplectic:

Let us take $X, Y$ two vector fields tangent to $E$ at $z \in E$; there is a unique $V_z$ such that $z \in V_z$, and two vector fields $X', Y'$ on $V_z$ verifying

$$i_{X'} X'_z = X_z,$$  

$$i_{X'} Y'_z = Y_z. \tag{1}$$  

$$i_{Y'} X'_z = X_z,$$  

$$i_{Y'} Y'_z = Y_z. \tag{2}$$

$^1$ The greatest integer not larger than $f(\theta)$ take the values 0,1,2,3 which remind us the strangeness quantum-numbers.

$^2$ If we consider $G$ the restricted Poincaré group, $U$ an orbit of $G$ in its coalgebra, and $M$ the space–time manifold, it was seen in Ref. [7] that it is a seeded fibre bundle that allows to extend the results of the relativistic elementary particles of J. M. Souriau (see Ref. [10]) to general relativity.