An Iteration Property of Lindenmayerian Power Series

Juha Honkala
Department of Mathematics
University of Turku
20500 Turku, Finland

Abstract. We establish an iteration property of Lindenmayerian power series. As an application we derive results on the series generating power of L systems.

1 Introduction

Formal power series play an important role in many diverse areas of theoretical computer science and mathematics [1,9,10,14]. The classes of power series studied most often in connection with automata, grammars and languages are the rational and algebraic series. In language theory formal power series often provide a powerful tool for obtaining deep decidability results [9,14]. A brilliant example is the solution of the equivalence problem for finite deterministic multitape automata given by Harju and Karhumäki [3].

In [9] Kuich and Salomaa gave a power series approach to formal language theory by using an algebraic notion of convergence. In [6-8] Kuich generalized the Kleene theorem, the Parikh theorem and the equivalence between context-freeness and acceptance by pushdown automata to complete semirings.

The framework of [9] was used in [4] to define Lindenmayerian power series, i.e., series obtained by morphic iteration. These series are generated by suitably modified L systems. We give a simple example.

Suppose A is a semiring and \( \Sigma \) is a finite alphabet. Denote the semiring of formal polynomials over \( \Sigma \) with coefficients in \( A \) by \( A < \Sigma^* > \) and assume that \( h : A < \Sigma^* > \rightarrow A < \Sigma^* > \) is a semiring morphism. Such a morphism necessarily satisfies \( h(\lambda) = \lambda \). We suppose also that \( h(a \cdot \lambda) = a \cdot \lambda \) holds for every \( a \in A \). Finally, assume \( \omega \in A < \Sigma^* > \). Now define the sequence \( r^{(i)}(i \geq 0) \) by \( r^{(0)} = \omega, r^{(i+1)} = h(r^{(i)}) \). Then \( \lim r^{(i)} \), if it exists, is a morphically generated series. Of course, we have to specify the convergence used in the limit process. In our work we allow also more complicated iteration. Instead of \( r^{(i+1)} = h(r^{(i)}) \) we might have, e.g., \( r^{(i+1)} = \alpha h_1(r^{(i)}) + h_2(r^{(i)})h_3(r^{(i)}) \), where \( \alpha \) is a letter and \( h_1, h_2, h_3 \) are, not necessarily distinct, morphisms of \( A < \Sigma^* > \).

In this paper we continue the study of Lindenmayerian series by establishing a basic iteration property of these series. By the well known iteration lemma for regular languages, if \( L \subseteq \Sigma^* \) is an infinite regular language, there exist words \( u, v, w \in \Sigma^* \) such that
Hence, we can repeat the nonempty subword \( w \) an arbitrary number of times provided that we prefix the resulting word by \( u \) and suffix it by \( v \). The analogous result for Lindenmayerian series can be stated as follows. Suppose \( r \) is a Lindenmayerian series with an infinite support. Then there exist morphisms \( h \) and \( g \) and a word \( w \) such that

\[
\{h(g^n(w)) \mid n \geq 0\} \subseteq \text{supp}(r)
\]

and the left-hand side is infinite. Notice that the application of \( h \) corresponds to the words \( u \) and \( v \) used above. This iteration property can be utilized in the study of Lindenmayerian series in much the same way as the pumping lemma is used in the study of regularity, see [13]. In particular, earlier results about combinatorial properties of HD0L languages can be utilized to give examples of series which are not Lindenmayerian.

For the motivation and background of our work, we refer to [4,5].

2 Definitions

We assume that the reader is familiar with the basic notions concerning semirings and formal power series (see [9]). For completeness, we specify the following.

The semiring of nonnegative real numbers is denoted by \( \mathbb{R}_+ \).

If \( A \) is a semiring and \( \Sigma \) is an alphabet, not necessarily finite, the \emph{semiring of formal power series with coefficients in} \( A \) \emph{and (noncommuting) variables in} \( \Sigma \) \emph{is denoted by} \( A << \Sigma^* >> \). If \( r \in A << \Sigma^* >> \) we denote

\[
r = \sum_{w \in \Sigma^*} (r, w)w \quad \text{and} \quad \text{supp}(r) = \{w \mid (r, w) \neq 0\}
\]

The set \( \text{supp}(r) \) is called the \emph{support} of \( r \). The subsemiring of \( A << \Sigma^* >> \) consisting of the series having a finite support is denoted by \( A < \Sigma^* > \). The elements of \( A < \Sigma^* > \) are referred to as \emph{polynomials}.

If \( L \subseteq \Sigma^* \) is a language, the series \( \text{char}(L) \in A << \Sigma^* >> \) is defined by

\[
\text{char}(L) = \sum_{w \in L} w
\]

and called the \emph{characteristic series} of \( L \). If \( r, s \in A << \Sigma^* >> \), the \emph{Hadamard product} \( r \odot s \in A << \Sigma^* >> \) of \( r \) and \( s \) is defined by

\[
(r \odot s, w) = (r, w)(s, w), \quad w \in \Sigma^*.
\]

In the sequel we need a notion of convergence. We follow [9].

A \emph{sequence} in \( A \) is a mapping \( \alpha : \mathbb{N} \rightarrow A \). The set of all sequences in \( A \) is denoted by \( A^\mathbb{N} \). If \( \alpha \in A^\mathbb{N} \), we denote \( \alpha = (\alpha(n)) \). For \( \alpha \in A^\mathbb{N} \) and