A Boltzmann Taylor Galerkin FEM for Compressible Euler Equations

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Abstract: A new Taylor Galerkin approach for compressible Euler equations based on the Kinetic Theory of Gases has been presented. The new Boltzmann Taylor Galerkin (BTG) FEM has been shown to be as accurate and as fast as the two step Taylor Galerkin and less dissipative than other Boltzmann schemes for continuum gas dynamics.

1 Introduction

Recent years have seen a remarkable progress in the solution of convection dominated problems by the finite element method. The Taylor Galerkin FEM of Donea[1] and especially its two step version of Swansea group[2] and the Streamline Upwind Petrov Galerkin (SUPG) method of Hughes [3] have been applied to a variety of problems ranging from simple convection equation to the compressible Navier-Stokes equations of fluid dynamics. Yet there is no agreement about the optimal finite element formulation for hyperbolic system of conservation laws which admit discontinuous solutions such as shocks and contact discontinuities. In this paper, a new line of research for developing novel finite element method for compressible Euler equations is put forward. The method exploits the rich connection between the Boltzmann equation of Kinetic Theory of Gases and the equations of continuum gas dynamics. This new method which we call as the Boltzmann Taylor Galerkin FEM is based on the well known fact that the Euler equations are moments of the Boltzmann equation when the velocity distribution is a Maxwellian.

2 Basic Theory of Boltzmann Schemes

Let us illustrate the basic idea with reference to one dimensional unsteady Euler equations

\[
\frac{\partial U}{\partial t} + \frac{\partial G}{\partial x} = 0
\] (1)

The vector of conserved variables \( U \) and flux vector \( G \) are given by \( U = [\rho, \rho u, \rho e]^T \) and \( G = [\rho u, \rho u^2, (\rho e + p) u]^T \), where \( \rho \) = mass density, \( u \) = fluid velocity, \( p \) = pressure, \( e \) = total specific energy given by \( e = \frac{p}{\gamma (\gamma - 1)} + \frac{1}{2} u^2 \). Equation (1) can be
obtained as the $\Psi$ moment of the 1-D Boltzmann equation

$$\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} = 0$$  

(2)

where $F$ is the Maxwellian velocity distribution given by,

$$F = \frac{p}{I_0} \sqrt{\frac{\beta}{\pi}} \exp \left[ -\beta(v - u)^2 - \frac{I}{I_0} \right]$$  

(3)

here $\beta = \frac{1}{2RT}$, $R$ = gas constant per unit mass, $v$=molecular velocity, $I$ = internal energy variable corresponding to non-translational degrees of freedom and $I_0$ is defined as $I_0 = \frac{3-\gamma}{4(\gamma-1)\beta}$. The moment function vector $\psi$ is defined by $\psi = \left[ 1, v, I, \frac{v^2}{2} \right]^T$ and is precisely the vector of fundamental collisional invariants. The Euler equations (1) can then be cast in the compact form

$$\langle \psi, \frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} \rangle = 0$$  

(4)

where the scalar product $\langle \psi, f \rangle$ is defined by

$$\langle \psi, f \rangle = \int_{-\infty}^{\infty} dI \int_{-\infty}^{\infty} dv \psi f(v)$$  

(5)

We thus see that Euler equations (which are a system of nonlinear hyperbolic system of conservation laws) are the moments of Boltzmann equation without the collision term which is a linear and scalar hyperbolic equation.

### 3 The Boltzmann Taylor Galerkin Scheme

We now proceed to develop the BTG scheme scheme. The various steps in the formulation are as follows.

**Step 1:** We start with the Lax-Wendroff step for the Boltzmann eqn.(2).

$$F^{n+1} = F^n + \Delta t F^n + \frac{\Delta t^2}{2} F^n_{tt}$$  

(6)

$$F^{n+1} = F^n - \Delta t (vF)_x + \frac{\Delta t^2}{2} (v^2 F)_{xx}$$  

(7)

**Step 2:** Now we apply the standard Galerkin FEM to equation (7) to obtain

$$\int_0^L (F^{n+1} - F^n) W dx = -\Delta t \int_0^L W (vF)_x dx + \frac{\Delta t^2}{2} \int_0^L W (v^2 F)_{xx} dx$$  

(8)

where $W$ is the weighting function, $L$ is the length of the domain. Integrating by parts the last term in eqn.(8), we obtain

$$\int_0^L W \Delta F^{n+1} dx = -\Delta t \int_0^L W (vF)_x dx - \frac{\Delta t^2}{2} \int_0^L \frac{\partial W}{\partial x} \frac{\partial (v^2 F)}{\partial x} dx + R_b$$  

(9)