A new second order positivity preserving kinetic schemes for the compressible Euler equations

Jean-Luc Estivalezes\(^1\), Philippe Villedieu\(^2\)

\(^1\)ONERA-CERT, 2 Av Edouard Belin, 31055 Toulouse cedex, France
\(^2\)MIP, Unité mixte CNRS , Université de Toulouse III
118 Route de Narbonne, 31062 Toulouse cedex, France

Abstract: We present a new second order kinetic flux-splitting schemes for the compressible Euler equations and we prove that this scheme is positivity preserving (i.e \(\rho\) and \(T\) remain \(\geq 0\)). Our first order kinetic scheme is based on the Maxwellian equilibrium function and was initially proposed by Pullin. Our higher order extension can be seen as a variant of the so called corrected anti-diffusive flux approach. The necessity of a limitation on the antidiffusive correction appears naturally in order to satisfy the constraint of positivity.

1 Introduction

In this paper, we present a new theoretical way to construct a second order Kinetic Flux Vector Splitting scheme for the Euler equations. Different approaches have already been proposed in Deshpande, Perthame, Prendergast. The interesting point here is that density and pressure can be proved to remain non-negative under a CFL-like condition. Higher order extensions following the same approach can be easily performed with the same properties.

Here, for the sake of simplicity, we will only give a kinetic interpretation of our scheme. The mathematical tools which enable to prove the non-negativity of density and pressure are given in Estivalezes and Villedieu.

The paper is organized as follow. First we recall the main features of the first order scheme, then we show how to build a truly second order KFVS scheme and we give the main theoretical results. Lastly numerical results for various 1D test cases (Sod test, Blast Waves interactions, Vacuum apparition) are shown.

2 First order scheme

For the sake of simplicity, we only consider the case of the one dimensional Euler equations for a gas with one degree of freedom \((\gamma = 3)\). It is well known that the Euler equations are the \(K\)-moment of the Boltzmann equation when the distribution function is Maxwellian, \(K\) being the collision vector of components \((1, \xi, \xi^2/2)\).

The first order scheme is constructed in two step like in Mazet.
First the free kinetic transport equation is solved for each $\xi$ by the explicit Courant scheme:

$$f_{n+1}^{n} = f_{n}^{n} - \lambda \max(\xi, 0)(f_{n+1}^{n} - f_{n}^{n}) - \lambda \min(\xi, 0)(f_{n}^{n} - f_{n-1}^{n})$$

where $\lambda = \Delta t / \Delta x$, $f_{n}^{n}$ is the Maxwellian distribution associated to the macroscopic state of the gas at the time $t_{n}$, and $\xi$ is the microscopic velocity.

$$f_{n}^{n}(\xi) = \frac{\rho_{n}^{n}}{\sqrt{2\pi T_{n}^{n}}} \exp \frac{-(\xi - U_{n}^{n})^{2}}{2T_{n}^{n}}$$

Where $\rho_{n}^{n}$, $\rho_{n}^{n}U_{n}^{n}$ and $T_{n}^{n}$ are respectively cells values of the density, the macroscopic velocity and the temperature.

The second step consists in taking the $K$-moment of (1). We get the following Kinetic Flux Splitting scheme for the Euler equations:

$$w_{i}^{n+1} = w_{i}^{n} - \lambda(g_{i+1/2} - g_{i-1/2})$$

where:

$$w_{i}^{n} = (\rho_{i}^{n}, \rho_{i}^{n}U_{i}^{n}, E_{i}^{n}) \quad g_{i+1/2} = F^{+}(w_{i+1}^{n}) + F^{-}(w_{i+1}^{n})$$

and

$$F^{+}(w_{i}^{n}) = \int_{0}^{\infty} \xi K(\xi)f_{i}^{n}(\xi)d\xi$$

The entropy consistency and the non-negativity of this scheme can be proved under a CFL like condition (Villedieu and Mazet).

3 Second order scheme

The main idea of our approach is to use a second order scheme at the kinetic level. Our extension can be derived in three steps:

First (1) is replaced by a second order scheme:

$$f_{i}^{n+1} = f_{i}^{n} - \lambda \max(\xi, 0)(f_{i-1/2,R}^{n+1/2} - f_{i-1/2,L}^{n+1/2}) - \lambda \min(\xi, 0)(f_{i+1/2,R}^{n+1/2} - f_{i+1/2,L}^{n+1/2})$$

where $f_{i+1/2,R}^{n+1/2}$ and $f_{i-1/2,L}^{n+1/2}$ are respectively right and left second order estimates of $f$ at the node $i + 1/2$. For example, we have:

$$f_{i+1/2,L}^{n+1/2} = f_{i}^{n} + \Delta f_{i}^{n} \quad \text{with} \quad \Delta f_{i}^{n} = p_{i}^{n}\Delta x/2 + q_{i}^{n}\Delta t/2$$

$p_{i}^{n}$ and $q_{i}^{n}$ being respectively second order estimates of $(\partial f / \partial x)_{i}^{n}$ and $(\partial f / \partial t)_{i}^{n}$.

Note that (2) is equivalent to:

$$f_{i}^{n} = C \exp \frac{-1}{r} \left[ s_{i}^{n} + \frac{(\xi - U_{i}^{n})^{2}}{2T_{i}^{n}} \right]$$

Where $s_{i}^{n} = -r \log p_{i}^{n} + 1/2r \log T_{i}^{n}$ is the specific entropy. So by (6) we get the following expressions for $p_{i}^{n}$ and $q_{i}^{n}$:

$$p_{i}^{n} = f_{i}^{n} \left( -\frac{p_{s}}{r} + \frac{p_{u}}{rT_{i}^{n}}(\xi - U_{i}^{n}) + \frac{p_{r}}{2rT_{i}^{n}}(\xi - U_{i}^{n})^{2} \right)$$