Concepts on Boundary Conditions in Numerical Fluid Dynamics

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1 Introduction

Boundary conditions are important in numerical fluid dynamics — this is a matter of course, though, the methods are not pursued deeply in most problems such as the flow around a body (bodies) or in a cavity, since there seems no more than putting the normal velocity zero on a wall, or putting the tangential velocity zero too, if the wall is non-slip. In some problems, boundary conditions are essential — fluids with free surface or density interface, fluids around a moving or oscillating body (bodies), open boundary, and so on. For successful numerical experiment of these problems, boundary conditions with physical significance are required, instead of expedient ones.

2 Deformable Cell and Laws of Conservation

In the present method of numerical analysis of fluids [1], the fluid in the domain of analysis is divided into deformable cells according to the movable boundaries. Since the shape of each cell differs from one to one, and it deforms, equations for numerical analysis are derived from integral-type laws of conservation here, instead of the hydrodynamic equations finite-differenced. When the fluid is incompressible, the law of conservation of volume (LCV)

\[ 0 = \left( \int dS \cdot \mathbf{v} \right)_{v}^{n+1} = \Delta V(q_{n+1}) \]

must be satisfied concerning a small domain around a vertex subscripted as \( v = 1, 2, \cdots \) (LCV domain, see Fig. 1) not only within the domain but also adjacent to boundaries and in corners at the time step \( n + 1 \), where \( \mathbf{v} \) is the fluid velocity, and the surface integral covers all the surfaces of the LCV domain. In two-dimensional case, this is rewritten as a linear combination of the flux \( q = b \mathbf{v} \) (b: the thickness of the fluid), which \( \Delta V(q_{n+1}) \) shows.

Treatment of boundaries can be described conveniently on the basis of the LCV adjacent to a boundary or in a corner. The flux \( q_{A} \) on a boundary or at a corner is expressed by the normal flux \( q_{n} \) and the tangential flux \( q_{s} \) as \( q_{A} = (q_{n} n + q_{s} s)_{A} \),

\[ q_{A} = [(q_{n})(2)^{s}(1) - (q_{n})(1)^{s}(2)]/(s(1) \times s(2)) \cdot z \]

where \( n, s \) are the normal and the tangential vector on the boundary, \( (1), (2) \) denote two boundaries crossing at the
corner, and \( z \) is the unit vector normal to the domain of analysis. The terms without \( q_n \) in \( \Delta V \) are collected into \( \Delta V_{B,C} \), and each \((q_s)^{n+1}\) is expressed linear in \( q_n^{n+1} \) within the domain as \((q_s)^{n+1}_A = (q_n)^{n+1}_A + K_A \cdot q_n^{n+1} \) as for most tangential boundary conditions. Then, the LCV is rewritten as

\[
0 = (q_n\Delta s)^{n+1}_s + \Delta V_{B}(\{q^{n+1}\}),
\]

(2)

\[
0 = (q_n\Delta s)^{n+1}_s + (q_n\Delta s)^{n+1}_s + \Delta V_{C}(\{q^{n+1}\}),
\]

(3)

where \( \Delta s_{5,(1),(2)} \) are the contact length between the LCV domain and the boundary. This formulation removes an instability concerning tangential boundary conditions such as free slip wall when \((q_s)^n\) is used expediently, instead of \((q_s)^{n+1}\).

3 Volume Force and Surface Force

There are fluids which react to an external field: dielectric fluid under an electric field or magnetic fluid under a magnetic field. In the equation of motion considering these fluids also, we find the gravity force, the pressure force, the viscous force and the electric or magnetic force. In physical sense, the forces which tend to zero together with the density are called volume force, and the others are surface force. In numerical analysis of incompressible fluids, however, these terms could be used in another way.

The momentum \( Mv \) in a cell \( c = A, B, \ldots \) is renewed between the time steps \( n \) and \( n + 1 \) with the increment \( \Delta t \), by adding to it the convective flux including the difference of the fluid velocity \( v \) and the deformation velocity of the surface of the cell \( u \), and the aforementioned forces. This is shown, equivalent to the equation of motion, in two equations as

\[
(Mv)^{(n)}_c = (Mv)^{(n)}_c + \int \mathbf{dS} \cdot (u - v)\rho v + \mathbf{V} \] \( \Delta t \),

(4)

\[
(Mv)^{(n+1)}_c = (Mv)^{(n)}_c - \int \int \int \mathbf{dV} \nabla p^* \] \( \Delta t \),

(5)

where \( \rho \) and \( V \) are the fluid density and the viscosity stress tensor.

After the momentum in each cell is renewed, the fluxes from these momenta must satisfy the LCV at \( n + 1 \). The procedure to determine the pressure on vertices answering this requirement intervenes between the procedures to add the forces without the pressure force and the pressure force itself. Though the pressure force is evidently the surface force, parts of the gravity force or the electric or magnetic force can be transferred to the pressure force to improve the accuracy of the numerical analysis. Then, the surface force \( -\nabla p^* \) is newly defined, and the sum of the other forces results in the volume force \( F^* \).