Vortex Methods for Three-Dimensional Separated Flows

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Abstract: Traditionally, vortex methods have been used to model unsteady, high Reynolds number incompressible flow by representing the fluctuating vorticity field with a few tens to a few thousand Lagrangian elements of vorticity. Now, with the advent of fast vortex algorithms, bringing the operating count per timestep down to O(N) from O(N^2) for N computational elements, and recent developments for the accurate treatment of viscous effects, one can use vortex methods for high resolution simulations of the Navier-Stokes equations. Their classical advantages still hold – (1) computational elements are needed only where the vorticity is nonzero (2) the flow domain is grid free (3) rigorous treatment of the boundary conditions at infinity is a natural byproduct and (4) physical insights gained by dealing directly with the vorticity field – so that vortex methods have become an interesting alternative to finite difference and spectral methods for unsteady separated flows.

1 Basic Equations

To describe vortex methods we start with the three-dimensional vorticity equation for the vorticity field $\omega$ in a constant density, incompressible flow,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \nabla^2 \omega .$$

(1)

where $\nu$ is the kinematic viscosity. Combining the incompressibility condition for the velocity field $u$, $\nabla \cdot u = 0$, and the definition of vorticity, $\nabla \times u = \omega$, we find that

$$\nabla^2 u = -\nabla \times \omega$$

(2)

We will consider vortex methods in the context of bluff-body flows and so we are interested in solutions to (1) and (2) corresponding to the no-slip at the surface of a rigid body moving with velocity $U_b$,

$$u|_{\text{surface}} = U_b|_{\text{surface}}$$

(3)
and free stream conditions at infinity

\[ u \rightarrow U_\infty(t) \text{ as } |x| \rightarrow \infty \] (4)

To simplify the description of the method we restrict the discussion to non-rotating bodies. For the treatment of rotating bodies see Koumoutsakos (1993). The solution to (2) satisfying (3) and (4) is given in terms of the infinite medium Green's function

\[ u(x) = -\frac{1}{4\pi} \int \frac{(x - x') \times \omega(x') \, dx'}{|x - x'|^3} + U_\infty(t) \] (5)

In our numerical approach described below it is important to recognize that the no-slip boundary condition is maintained by a continual flux of vorticity from the body surface into the fluid. Mathematically this flux is such that \( \omega \) is nonzero only in the fluid (i.e., external to the body) and that (5) gives the result that

\[ \omega \cdot t \bigg|_{\text{surface}} = U_b \cdot t \bigg|_{\text{surface}} \] (6)

where \( t \) is any tangent vector at the body surface. That (5) and (6) imply (3) may be shown as follows. Let \( \psi \), the stream function for the imaginary fluid within the body, be given by

\[ \psi = \frac{U_b \times x}{2} + \psi' \] (7)

Thus within the body \( B \)

\[ u = \nabla \times \psi = U_b + \nabla \times \psi' \] (8)

and

\[ \nabla^2 \psi = \nabla^2 \psi' = -\omega = 0 \] (9)

because all vorticity is external to \( B \). Now we use Green's identity

\[ \int_B [(\nabla u) \cdot (\nabla v) + u \, \nabla^2 v] \, dx + \int_{\partial B} u \frac{\partial v}{\partial n} \, ds = 0 \] (10)

with \( u = v = \psi_i' \). Here \( n \) is in the outward normal direction. From (6) and (8)

\[ (\nabla \times \psi') \cdot t = 0 \] (11)

at every point on the surface \( \partial B \) for any tangent vector \( t \). Thus, \( \frac{\partial \psi'}{\partial n} = 0 \) so that (10) reduces to

\[ \int_B |\nabla \psi'_i|^2 \, dx = 0 \quad (i = 1, 2, 3) \] (12)

Hence \( \psi' = \text{const} \) and

\[ u = U_b \text{ in } B. \] (13)