The Interval Order Polytope of a Digraph *

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Abstract. We introduce the interval order polytope of a digraph D as the convex hull of interval order inducing arc subsets of D. Two general schemes for producing valid inequalities are presented. These schemes have been used implicitly for several polytopes and they are applied here to the interval order polytope. It is shown that almost all known classes of valid inequalities of the linear ordering polytope can be explained by the two classes derived from these schemes. We provide two applications of the interval order polytope to combinatorial optimization problems for which to our knowledge no polyhedral descriptions have been given so far. One of them is related to analyzing DNA subsequences.

1 Introduction

Interval orders and their cocomparability graphs, the interval graphs, play an important role not only in the theory of partially ordered sets and graph theory (cf., e.g., [Fis85]) but also for combinatorial optimization problems. This is due to the fact that each element is associated with an interval, which may be interpreted as a time interval, for example, in scheduling [MR89], or as a subsequence in a linear sequence of items, for example, a subsequence of the sequence of bases of a DNA in molecular biology [GKS94], or a sequence of electronic units in linear VLSI layout styles [Möhl90, Mühl93a].

Since most of these optimization problems are $NP$-hard one would like to have good enumeration approaches to solve them exactly as well as a tool for gaining more structural insights. Both can be achieved by a polyhedral approach that is based on a "good" (partial) polyhedral description of the solution space of the combinatorial optimization problem.

This paper initializes a basis for such an approach for problems related to interval orders and interval graphs. It introduces the interval order polytope of...
a digraph and provides some basic results on this polytope (Sect. 3). It shows how to construct two classes of stronger valid inequalities thereby providing two general schemes implicitly used in the literature. For one of these classes the separation problem can be solved in polynomial time (Sect. 4). For the *linear ordering polytope*, which has achieved considerable interest in the last decade, it offers new interpretations for all but one classes of known valid inequalities, implying a polynomial separation algorithm for the Möbius Ladder inequalities (Sect. 5). Finally, it illustrates two applications of the interval order polytope (Sect. 6).

2 Preliminaries

A *directed graph (digraph)* $D = (V, A)$ is a graph with finitely many nodes and (directed) arcs. Loops and multiple arcs are not allowed, but arcs connecting the same nodes in different directions are possible.

A *partially ordered set (poset)*, denoted by $P = (V, <)$, is a finite set $V$ together with a transitive, irreflexive relation $<$ on $V$. We represent a poset by a digraph $D = (V, A)$ with $(i, j) \in A$ iff $i < j$, and say that $D$ is a poset or $A$ is a poset.

An *interval order* is a poset $P = (V, <)$ such that there exists a real interval $[l_v, r_v]$ for each $v \in V$ with the property $v < w$ iff $r_v \leq l_w$. Such a set of intervals is called an *interval representation* of $P$. In addition, we will use the following characterization of interval orders. We denote by $2+2$ a digraph with mutually distinct nodes $i, j, k, l$, and arcs $(i, j)$ and $(k, l)$ only.

**Theorem 1 (Interval order characterization [Fis85])**. Let $D = (V, A)$ be a poset. Then the following conditions are equivalent:

1. $D$ is an interval order,
2. $D$ does not contain a $2+2$ as an induced subdigraph.

Given a subset $B$ of the arcs of the digraph $D = (V, A)$ we denote by $x^B$ the corresponding incidence (or characteristic) vector in $\mathbb{R}^A$, i.e., $x^B_a = 1$ if $a \in B$ and $x^B_a = 0$, otherwise.

3 The Interval Order Polytope

Given a digraph $D = (V, A)$ we define the *interval order polytope* as

$\mathcal{P}_{IO}(D) := \text{conv}\{x^B \mid B \subseteq A, B \text{ is an interval order}\}$.

If no confusion can arise, we often use $\mathcal{P}_{IO}$ instead of $\mathcal{P}_{IO}(D)$.

Since every single arc in $D$ as well as the empty set of arcs describes an interval order the dimension of $\mathcal{P}_{IO}(D)$ is equal to $|A|$, i.e., $\mathcal{P}_{IO}(D)$ is full-dimensional.

Before presenting an initial linear relaxation of $\mathcal{P}_{IO}$ each integer solution of which already defines an interval order, we provide a useful lifting theorem.