Combining Semidefinite and Polyhedral Relaxations for Integer Programs

C. Helmberg[1], S. Poljak[2], F. Rendl[1], and H. Wolkowicz[3]

[1] Technische Universität Graz, Institut für Mathematik, Kopernikusgasse 24, A-8010 Graz, Austria

[2] Universität Passau, Institut für Mathematik und Informatik, Innstraße 33, 94030 Passau, Germany

[3] University of Waterloo, Department of Combinatorics and Optimization, Waterloo, Canada

Abstract. We present a general framework for designing semidefinite relaxations for constrained 0-1 quadratic programming and show how valid inequalities of the cut-polytope can be used to strengthen these relaxations. As examples we improve the ϑ-function and give a semidefinite relaxation for the quadratic knapsack problem. The practical value of this approach is supported by numerical experiments which make use of the recent development of efficient interior point codes for semidefinite programming.

Key words: integer linear programming, semidefinite programming, quadratic 0-1 optimization, interior point methods.

1 Introduction

Semidefinite Programming offers excellent possibilities for the design of very tight relaxations for combinatorial problems. With the development of efficient interior point methods [10] it is now possible to solve these relaxations in reasonable time. Indeed, for some problems the semidefinite approach is already more efficient than sophisticated linear programming codes [9]. We present a general framework for designing semidefinite relaxations for constrained 0-1 quadratic programming based on ideas of Lovász et al [15] and Balas et al [2]. We show that these relaxations can be considerably strengthened by exploiting the equivalence of the 0-1 semidefinite relaxation for 0-1 quadratic programming and the (-1,1) semidefinite relaxation for max-cut. We believe that this is the first step towards the development of a new powerful tool for solving difficult subproblems in integer programming.

Probably the best known semidefinite relaxation is the Lovász ϑ-function for the maximum clique problem [14]. The potential of this relaxation was recognized immediately, but since no efficient semidefinite programming codes were available at that time it was more or less impossible to solve it computationally. In the sequel semidefinite formulations were — as far as practical applications

are concerned — mainly used to construct new valid inequalities for linear programs. Combinatorial applications were also an important inducement for the development of interior point algorithms for semidefinite programming [1, 10]. Recently several results confirm the usefulness of semidefinite programming in combinatorial optimization. Based on the semidefinite formulation of the max-cut upper bound $\varphi(G)$ introduced in [6] Goemans and Williamson developed an 0.878 approximation algorithm for max-cut, thereby proving a worst case error of 14% for $\varphi(G)$. It was noted by Laurent and Poljak [12] that there is a close relationship between the max-cut polytope and the feasible set of the semidefinite relaxation $\varphi(G)$. Computationally this connection is exploited in [9] where $\varphi(G)$ is considerably improved by an interior point cutting plane approach using the clique facets of the max-cut polytope.

The max-cut problem is well known to be equivalent to 0-1 quadratic programming. However, we are interested in the more general case of constrained quadratic programming,

$$\text{(CQP)} \qquad \begin{array}{ll} \text{Maximize} & x^tCx \\ \text{subject to} & Ax \leq b \\ & x^tA_ix \leq a_i \qquad i = 1 \dots k \\ & x \in \{0,1\}^n , \end{array}$$

with C being an arbitrary symmetric $n \times n$ matrix, A an $m \times n$ matrix, $b \in \Re^n$. The A_i are symmetric $n \times n$ matrices, and $a_i \in \Re$. Since $x_i^2 = x_i$, we may assume without loss of generality that there is no linear term in the cost function, or to put it differently, the linear term consists of the main diagonal of the matrix C. We will use ideas from Lovás et al. [15] and Balas et al. [2, 3, 4] to derive tractable relaxations of (CQP). It turns out that these relaxations have equivalent formulations in the max-cut setting. We can therefore exploit the properties of the cut-polytope for the 0-1 programming case, too.

To demonstrate the quality and potential of this approach we give computational results for two specific applications, the first being the maximum clique problem. Let $G(V, E)$ denote an undirected loopless graph on n vertices $V = \{1, \dots, n\}$ with edge set E. We want to find the maximum clique with respect to weights w_i on the nodes,

$$\text{(MCL)} \qquad \begin{array}{ll} \text{Maximize} & w^Tx \\ \text{subject to} & x_ix_j = 0 \qquad \forall (ij) \notin E \\ & x \in \{0,1\}^n . \end{array}$$

The semidefinite relaxation is indeed the ϑ-function, we will improve it further by adding triangle facets of the cut-polytope.

The second problem is the quadratic knapsack problem, i.e. quadratic programming with one inequality constraint,

$$\text{(QK)} \qquad \begin{array}{ll} \text{Maximize} & x^tCx \\ \text{subject to} & a^tx \leq b \\ & x \in \{0,1\}^n , a \in \Re^n, b \in \Re. \end{array}$$