On Maximal Spherical Codes I

Peter Boyvalenkov, Ivan Landgev

Institute of Mathematics,
Bulgarian Academy of Sciences,
8 G.Bonchev str., Sofia 1113, Bulgaria
e-mail: sectmoi@bgearn.bitnet

Abstract. We investigate the possibilities for attaining two Levenshtein upper bounds for spherical codes. We find the distance distributions of all codes meeting these bounds. Then we show that the fourth Levenshtein bound can be attained in some very special cases only. We prove that no codes with an irrational maximal scalar product meet the third Levenshtein bound. So in dimensions $3 \leq n \leq 100$ exactly seven codes are known to attain this bound and ten cases remain undecided. Moreover, the first two codes (in dimensions 5 and 6) are unique up to isometry. Nonexistence of maximal codes in all dimensions $n$ with cardinalities between $2n + 1$ and $2n + [7\sqrt{n}]$ is shown as well. We prove nonexistence of several infinite families of maximal codes whose maximal scalar product is rational. The distance distributions of the only known nontrivial infinite family of maximal codes (due to Levenshtein) are given.

1 Introduction

An $(n, M, s)$ spherical code is a finite subset $W$ of $n$-dimensional Euclidean sphere $S^{n-1}$ with cardinality $|W| = M$ and a maximal scalar product $s = \max\{(x, y) : x, y \in W, x \neq y\}$. The maximal possible cardinality of an $(n, M, s)$ code is

$$A(n, s) = \max\{|W| : W \text{ is an } (n, |W|, s) \text{ code }\}.$$

Any $(n, A(n, s), s)$ code is called maximal.

The numbers $A(n, s)$ are well known for $s \leq 0$ in all dimensions [11]. However, only several values with $n \geq 3$ and $s > 0$ have been found [1, 6, 9].

The best known bounds on $A(n, s)$ are obtained from linear programming by Levenshtein [7, 8]. In this paper we investigate the possibilities for attaining the following two Levenshtein bounds

$$A(n, s) \leq L_3(n, s) = \frac{n(1 - s)(2 + s + ns)}{1 - ns^2} \quad (1)$$

for $0 \leq s \leq \frac{1}{1 + \sqrt{n+3}}$, and

$$A(n, s) \leq L_4(n, s) = \frac{2n(1 - s)(1 + 2s + 2ns)}{1 + 2s - (n + 2)s^2} \quad (2)$$
for $\frac{1}{1+\sqrt{n+3}} \leq s \leq \frac{1}{\sqrt{n+2}}$.

To obtain his bounds Levenshtein uses suitable polynomials (which are extremal in some sense) of degrees 1, 2, ... The first two Levenshtein bounds (obtained by polynomials of degrees one and two) are in fact the Rankin bounds for $s \leq 0$ [11]. Here we consider the next two bounds ((1) and (2)) which we refer to as third and fourth respectively.

If $W$ is a spherical code and $x \in W$, then the distance distribution of $W$ with respect to $x$ is the system of nonnegative integer numbers $\{A_t(x) : -1 \leq t < 1\}$ where $A_t(x) = |\{y \in W : (x, y) = t\}|$. It turns out that the distance distributions of all codes attaining the Levenshtein bounds can be found by solving of a (Vandermonde) system [1, Theorem 7.4], [3, Theorem 2.1], [4, Theorem 2]. Moreover, the distance distributions do not depend on the point $x$. So we shall write $A_t$ instead of $A_t(x)$.

In what follows we suppose that $n \geq 3$ and $s > 0$ are such that the number $L_3(n, s)$ (or $L_4(n, s)$ resp.) is integer. Then we compute the distance distributions of the putative $(n, L_3(n, s), s)$ $(n, L_4(n, s), s)$ resp.) maximal codes.

In Section 2 we find the distributions of all $(n, L_3(n, s), s)$ and $(n, L_4(n, s), s)$ codes. As an immediate consequence we show that $(n, L_4(n, s), s)$ maximal codes could exist only if $s = 1/(1+\sqrt{n+3})$ and $n+3$ is an odd square or $s = 1/\sqrt{n+2}$ and $n+2$ is an odd square (see (2)). In Section 3 we prove a Lloyd type theorem showing that no maximal $(n, L_3(n, s), s)$ codes with $s$ irrational exist. Then in Section 4.1 we investigate for maximal codes the dimensions $n \leq 100$. We prove that the only $(n, L_3(n, s), s)$ codes in dimensions $3 \leq n \leq 20$ are the $(5, 16, 1/5)$ and $(6, 27, 1/4)$ codes which are unique up to isometry [3, 5]. In dimensions $21 \leq n \leq 100$ there are five known maximal codes and ten undecided cases. In Section 4.2 we prove that no $(n, L_3(n, s), s)$ maximal codes with $2n+1 \leq L_3(n, s) \leq 2n+\lfloor\sqrt{n}\rfloor$ exist. Nonexistence of certain infinite families of $(n, L_3(n, s), s)$ codes with $s$ rational is shown in Section 5. In particular, we prove that if $n \neq 5$ then $A(n, 1/n) \leq 3n$ instead of $3n+1$ by (1). We list some (infinitely many) undecided cases that come from tight spherical 4-designs [3].

2 Computing Distance Distributions of All $(n, L_3(n, s), s)$ and $(n, L_4(n, s), s)$ Codes

Let $W$ be an $(n, L_3(n, s), s)$ code with $s > 0$ (since $A(n, 0) = L_3(n, 0) = 2n$ [11]). We set $s = 1/m$ and then

$$L_3(n, s) = 2n + \alpha = \frac{n(m-1)(2m+n+1)}{m^2 - n}$$

where $\alpha$ is integer and $m \geq 1 + \sqrt{n+3}$. By [8, Theorem 4.1], [4, Theorem 1] $A_t > 0$ is possible only for

$$t \in \left\{ \frac{1}{m}, -\frac{m+1}{m+n} \right\}.$$