Large–Scale Structure of the Universe

Asymptotics of Burgers' Turbulence

with Heavy–tailed Dependent Data

Yiming Hu\textsuperscript{2} and W.A. Woyczynski\textsuperscript{1,2}

\textsuperscript{1} Department of Statistics
\textsuperscript{2} Center for Stochastic and Chaotic Processes in Science and Technology
Case Western Reserve University
Cleveland, Ohio 44106

Abstract: Large-time asymptotics of the statistical solutions $v(t, x)$ of the Burgers' equation $v_t + vv_x = v v_{xx}$ is considered. The initial velocity potential is assumed to be of the shot noise type with dependent amplitudes and heavy tails. The problem arises naturally in the adhesion model of the large-scale distribution of matter in the Universe. As $t \to \infty$, the random field $v(t, x)$ becomes stochastically relatively asymptotically equivalent to a field with "saw-tooth" trajectories. The intermittent shocks of the velocity field correspond then to the regions of high density in the coupled passive tracer density field. This paper extends a result of S. Albeverio, S.A. Molchanov and D. Surgailis (1992), obtained for the case of independent amplitudes.

1 Introduction

It is a well known observational fact that the matter in the Universe is distributed in clusters and superclusters of galaxies, with giant voids between them. At this late epoch of the formation of the large scale structure,

- the dark (nonluminous) matter dominates;
- it acts as collisionless dustlike particles;
- no pressure effects need to be taken into account, with the Newtonian gravity being the only force of consequence;
- the radiative and gas dynamics effects are short range.

Assuming the flat, expanding universe, with the scale factor

$$a(t) = t^{2/3},$$

and the mean density

$$\bar{\rho} \propto a^{-3},$$

the evolution of the matter density $\rho = \rho(x, t)$ is usually (see e.g. Peebles (1980), Kofman et al. (1992)) described by the system of three coupled partial differential equations

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \nabla \cdot (\rho v) = 0,$$  \hspace{1cm} (1.1)
\[ \frac{dv}{dt} + Hv = -\frac{1}{a} \nabla \varphi, \]  
(1.2)

\[ \nabla^2 \varphi = 4\pi Ga^2 (\rho - \bar{\rho}), \]  
(1.3)

where \( v \) is the local velocity, \( \varphi \) is the gravitational potential, and \( H \) and \( G \) are, respectively, the Hubble and the gravitational constants. The three equations are, of course, the continuity equation, the Euler equation and the Poisson equation.

This system is not easy to analyze and several attempts have been made at simplifying it, while preserving the predictive ability of the reduced models. Introducing the velocity \( u = dx/da \) in the coordinates comoving with the expanding Universe, the Euler equation (1.2) is transformed into equation

\[ \frac{\partial u}{\partial a} + (u \cdot \nabla)u = -\frac{3}{2a} (u + \Lambda \nabla \varphi), \]  
(1.4)

with

\[ \Lambda = \left( \frac{3}{2} H^2 a^3 \right)^{-1} = \text{const}, \]

where the right-hand side represents, in the Lagrangian approach, the force acting on the particle. It is still a nonlocal operator so, in (1970), Zeldovich proposed a model in which it was assumed to be 0. This gives a clear Lagrangian picture as (1.4) becomes then the classical Riemann equation, which could explain formation of the pancake structures. This model has been adjusted by Gurbatov, Saichev and Shandarin (1985) (see also Shandarin and Zeldovich (1989), Weinberg, Gunn (1990)) who replaced the nonlocal term on the right-hand side of (1.4) by the Laplacian, to yield the Burgers equation

\[ \frac{\partial u}{\partial a} + (u \cdot \nabla)u = \nu \nabla^2 u, \]  
(1.5)

where the viscosity term is supposed to mimic the gravitation adhesion. The constant \( \nu \) should be small so that the viscosity effects do not affect the motion of the matter outside clusters. This adhesion model of the large structure of the Universe has been extensively studied in the astrophysical literature and satisfactorily tested against high resolution (512 \( \times \) 512) \( N \)-body simulations (Kofman et al (1992)).

In this paper we consider the Cauchy problem for the one-dimensional Burgers' equation

\[ v_t + vv_x = \nu v_{xx}, \]

with the shot noise initial velocity potential \( V(x) = \int_0^x v(0, y) dy \) of the form

\[ V(x) = \sum_{i=\infty}^{\infty} e^{\xi_i} \delta(x - x_i), \]  
(1.6)

where \( \{x_i\} \) is the standard Poisson point process on the real line and random variables

\[ \xi_i = \sum_{j=0}^{n} c_i \eta_{n+i-j}, \quad i \in \mathbb{N}, \]