Random Matrices of Circular Symplectic Ensemble

Karol Życzkowski

Instytut Fizyki im. M. Smoluchowskiego,
Uniwersytet Jagielloński, PL–30–059 Kraków, Poland

Abstract: Random unitary matrices of symplectic ensemble describe statistical properties of time-dependent, periodical quantum systems with a half-integer spin. We present a method of constructing random matrices typical to circular symplectic ensemble and show that the numerically generated unitary symplectic matrices display statistical properties of spectrum and eigenvectors according to the predictions of the random matrix theory.

1 Introduction

Random matrices, often used to describe statistical properties of complicated quantum systems with many degrees of freedom, are also applicable for simple quantum systems with few degrees of freedom, which exhibit chaos in the classical limit [1, 2]. A Hamiltonian of an autonomous quantum system may be represented by a Hermitian matrix of a Gaussian ensemble [3], whereas for a system periodically perturbed in time a more convenient characterization is provided by a unitary matrix representing the evolution operator propagating the wave function of the system over one period of the perturbation. Canonical ensembles of unitary matrices, invariant with respect to orthogonal, unitary or symplectic transformations where introduced by Dyson [4]. Such random matrices are also useful for investigating open scattering systems, described by a unitary S matrix [5].

Depending on the symmetry properties of the system one of the three canonical ensembles should be used in both cases. Systems possessing an antiunitary symmetry (mostly the time reversal invariance) display a linear repulsion of neighbouring energy levels (eigenphases) and are described by orthogonal ensembles. Unitary ensembles, appropriate for systems with the time reversal symmetry broken, exhibit quadratic level repulsion. Systems with a half-integer spin, a time-reversal invariance and no rotational symmetry pertain to the symplectic universality class, which is characterized by a quartic level repulsion [2]. Qualitatively speaking the presence of the Kramers degeneracy makes any additional accidental degeneracy very unlikely. This kind of spectral statistics was found
for a periodically kicked top with a half integer spin [6] and an appropriately modified version of the kicked rotator [7, 8]. Statistical properties of such time-dependent dynamical systems may be therefore described by random matrices of circular symplectic ensemble (CSE).

It is relatively easy to generate random Hermitian matrices pertaining to different universality classes - the matrix elements of such matrices are statistically independent random variables drawn according to a Gaussian distribution with zero mean. [3]. The only constrains are imposed by the algebraic conditions of symmetry (reality), hermiticity and symplecticity, involving pairs of elements.

Construction of unitary matrices typical of circular ensembles is more complicated, since unitarity imposes correlation between elements of the matrix. Recently we proposed a simple algorithm allowing one to construct random unitary matrices typical of circular unitary ensemble (CUE) and circular orthogonal ensemble (COE). In this work we present a method of generating random matrices characteristic of circular symplectic ensemble and show that obtained matrices conform to the predictions of random matrix theory (RMT).

2 Circular Unitary Ensemble

Circular unitary ensemble is defined by Haar measure in the space of $N \times N$ unitary matrices $U(N)$, invariant under the group of unitary transformations [4]. In order to construct numerically a unitary matrix typical of CUE we apply the parameterization of Hurwitz [9] and use the appropriate generalized Euler angles. An arbitrary unitary transformation $U$ can be composed from elementary unitary transformations in two-dimensional subspaces. The matrix of such an elementary unitary transformation will be denoted by $E^{(i,j)}(\phi, \psi, \chi)$. The only nonzero elements of $E^{(i,j)}$ are

$$
E_{kk}^{(i,j)} = 1, \quad k = 1, \ldots, N; \quad k \neq i, j \\
E_{ij}^{(i,j)} = \cos \phi e^{i\psi} \\
E_{ji}^{(i,j)} = -\sin \phi e^{-i\psi} \\
E_{jj}^{(i,j)} = \sin \phi e^{i\chi} \\
(1)
$$

From the above elementary unitary transformations one constructs the following $N - 1$ composite rotations

$$
E_1 = E^{(1,2)}(\phi_{12}, \psi_{12}, \chi_{12}) \\
E_2 = E^{(2,3)}(\phi_{23}, \psi_{23}, 0)E^{(1,3)}(\phi_{13}, \psi_{13}, \chi_{13}) \\
E_3 = E^{(3,4)}(\phi_{34}, \psi_{34}, 0)E^{(2,4)}(\phi_{24}, \psi_{24}, 0)E^{(1,4)}(\phi_{14}, \psi_{14}, \chi_{14}) \\
\ldots
$$

$$
E_{N-1} = E^{(N-1,N)}(\phi_{N-1,N}, \psi_{N-1,N}, 0)E^{(N-2,N)}(\phi_{N-2,N}, \psi_{N-2,N}, 0) \\
\ldots E^{(1,N)}(\phi_{1N}, \psi_{1N}, \chi_{1N})
$$

and eventually forms the unitary transformation $U$ as

$$
U = e^{i\alpha} E_1 E_2 E_3 \ldots E_{N-1}.
$$

(3)