1 Introduction: Why do we need Stochastic Center Manifolds?

In Section 4 of Boxler [2] in this volume we have seen that the Lyapunov exponents attached to a nonlinear cocycle that depends on a parameter \( \alpha \) are able to detect possible bifurcation points: a Lyapunov exponent vanishing at \( \alpha = \alpha_0 \) is a necessary condition for a stochastic bifurcation at \( \alpha_0 \). But we have also explained there that for the purpose of bifurcation theory it is not enough to consider only the linearized cocycle because it is just the nonlinear part of the system that is responsible for the existence of stable bifurcating solutions.

What do we know up to now about the behaviour of the entire (nonlinear) system? We have to distinguish between two situations:

1. Before having reached the stochastic bifurcation point:
   All the Lyapunov exponents are negative and hence the linearized system is asymptotically stable. Furthermore, Theorem 11 in [2] tells us that this behaviour carries over to the nonlinear system which is therefore asymptotically stable, too.

2. At the stochastic bifurcation point:
   The biggest Lyapunov exponent vanishes but all the other Lyapunov exponents remain negative. But then it is no longer possible to apply Corollary 7 in [2] and, even worse, we do not know anything about the relationship between the linearized and the original nonlinear cocycle.

The idea to deal with case 2 is now the following:

Since systems with negative Lyapunov exponents are well understood (because of their asymptotic stability) we ask whether at the bifurcation point the part of the cocycle corresponding to the negative Lyapunov exponents can be isolated. Its dynamical behaviour being clear it should then be possible to restrict the further analysis to the remaining part of the system where the Lyapunov exponent vanishes.

In fact, this heuristic idea can be made mathematically rigorous, and the device to do so is the concept of a stochastic center manifold. The idea of decomposing the cocycle with the help of stochastic center manifolds will also yield a considerable reduction of the dimension of the problem we will have to study if we are interested in bifurcation phenomena; very often it will be sufficient to deal with a one- or two-dimensional system although the original equation has been rather high dimensional.
2 Definition of Stochastic Center Manifolds

Let us assume the situation described in Section 3 of Boxler [2], i.e. our basic object is a nonlinear cocycle $\phi(t, \omega, x)$. In bifurcation theory it will depend on a parameter but for notational simplicity we will drop this dependence for the moment. Recall that we are allowed to assume without loss of generality that the cocycle has got a fixed point at 0. Furthermore, the multiplicative ergodic theorem applied to the linearized cocycle $\Psi(t, \omega)$ ensures the existence of $\tau \leq d$ real numbers $\lambda_1 > \ldots > \lambda_\tau$, the Lyapunov exponents, and of $\tau$ random linear subspaces $E_i(\omega) \subset \mathbb{R}^d$, the Oseledec spaces, which replace the real parts of the eigenvalues and the eigenspaces used in the deterministic case.

Being mainly interested in bifurcation theoretical applications we will assume that one Lyapunov exponent is zero (possibly with multiplicity $> 1$) and that the others are negative. For the general case, where positive exponents are considered as well, see Boxler [3].

In a first step we collect the Oseledec spaces corresponding to zero resp. negative Lyapunov exponents in order to obtain a new random coordinate system. Hence we define:

$$E_c(\omega) \overset{\text{def}}{=} \bigoplus_{\lambda_i = 0} E_i(\omega) \quad E_s(\omega) \overset{\text{def}}{=} \bigoplus_{\lambda_i < 0} E_i(\omega) \quad (1)$$

For all $t \in \mathbb{R}$ Oseledec's theorem will then yield the following decomposition:

$$\mathbb{R}^d = E_c(v_t \omega) \oplus E_s(v_t \omega) \quad (\text{recall that: } v_0 \omega = \omega) \quad (2)$$

Thus, for any $t$ we may decompose the cocycle into its components with respect to this coordinate system. The visualization of this projection at time 0 and at time $t$ might look as in Fig. 1.

Next we may decompose $\phi_c$ and $\phi_s$ into its linear and its nonlinear part; this yields the system which will be the starting point for our further investigations:

$$\phi_c(t, \omega, x_c, x_s) = \Psi_c(t, \omega)x_c + \Phi_c(t, \omega, x_c, x_s) \quad (3)$$
$$\phi_s(t, \omega, x_c, x_s) = \Psi_s(t, \omega)x_c + \Phi_s(t, \omega, x_c, x_s) \quad (4)$$

If these equations were decoupled it would be sufficient to examine (3) because the stability behaviour of (4) would be well-known thanks to Theorem 11 in [2]. The stochastic center manifold will enable us to carry out such a decoupling, i.e. to eliminate $x_s$ from equation (3) by replacing it by a function of $x_c$. How should such stochastic center manifolds be defined?

Forget about the nonlinear part of (3) and (4) for a moment. Then the equations are decoupled and we only have to study what happens in the directions of $E_c(\omega), E_s(\omega)$ resp. Thus, if there is a nonlinear part which is not too big (or, equivalently, if we restrict ourselves to a neighborhood of the origin) we will expect that locally, i.e. near the origin, the linearized system will be a good approximation and that we obtain the picture 2.