Maximal Extensions of Simplification Orderings

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Abstract. Several well-founded syntactic orderings have been proposed in the literature for proving the termination of rewrite systems. Recursive path orderings (RPO) and their extensions are the most widely used in theorem proving systems such as \textit{RRL, REVE, LP}. While these orderings can be total (up to equivalence) on ground terms, they are not \textbf{maximal}. That is, when used to compare non-ground terms, there can be terms such that for all ground substitutions, the first term is bigger than the second term, but these orderings declare the two terms as not comparable. A new family of orderings induced by precedence on function symbols, much like RPO, is developed in this paper. Terms are compared by comparing their paths. These ordering are shown to be maximal, and are hence called \textit{maximal path orderings}. The maximal extension of RPO can be defined using symbolic constraint solving procedures. Such a decision procedure can check, given two terms $s$ and $t$, whether there is a ground substitution $\sigma$ that makes $\sigma(s)$ bigger than $\sigma(t)$ using RPO. A new decision procedure for the existential fragment of ordering constraints expressed using RPO is given based on the idea of \textit{depth bounds}. It is shown that given two terms $s$ and $t$, if there is a ground substitution $\sigma$ which makes $\sigma(s)$ bigger than $\sigma(t)$ using RPO, then there is a ground substitution within depth $k \ast d + k$ which is also a solution, where $k$ is the number of variables in $s$ and $t$, and $d$ is the maximum of the depths of $s$ and $t$.

1 Introduction

Rewrite systems provide an interesting and useful model of computation based on the simple inference rule of "replacing equals by equals." Rewrite techniques have proved successful in many areas including theorem proving, specification and verification, and proof by induction.

The power of the rewriting approach stems from the ability to "orient" equality ($\leftrightarrow$), which is symmetric, into a directed "rewrite" relation ($\rightarrow$), which is

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anti-symmetric, using a "well-founded ordering." The rules are used for "simplifying" expressions by repeatedly replacing instances of left-hand sides by the corresponding right-hand sides. For example, the rules below express addition and multiplication over natural numbers.

\[
\begin{align*}
0 + x & \rightarrow x \\
\mathit{s}(x) + y & \rightarrow \mathit{s}(x + y) \\
0 \times x & \rightarrow 0 \\
\mathit{s}(x) \times y & \rightarrow y + (x \times y)
\end{align*}
\]

A sample derivation chain is \( s(0) \times s(0) \rightarrow s(0) + (0 \times s(0)) \rightarrow s(0) + 0 \rightarrow s(0 + 0) \rightarrow s(0) \).

Termination of such derivations is crucial for using rewriting in proofs and computations. Syntactic "path orderings" based on a precedence relation \( \succ_{f} \) on function symbols have been developed to prove termination of a set of rewrite rules. A comprehensive survey is [3].

The Recursive Path Ordering (RPO) [2] is the most commonly used ordering. When \( \succ_{f} \) is a total precedence, RPO is total (up to equivalence) on ground terms (terms without variables). That is, given two distinct ground terms, either they are equivalent under the ordering, or one of them is bigger.

A seemingly obvious way to extend RPO maximally to non-ground terms would be to say that \( s \) is bigger than \( t \) iff every ground instance of \( s \) is bigger than the corresponding ground instance of \( t \). This maximal extension of RPO is hard to define directly however. When RPO is lifted in a direct way to non-ground terms, it is not maximal (examples in Section 4). The paths ordering (KNSS) [6] or the recursive decomposition ordering (IRDS) [8] are attempts to extend RPO. See [12] for a survey of these extensions. These orderings are the same as RPO when comparing ground terms if the precedence \( \succ_{f} \) is total, but they strictly include RPO when applied to non-ground terms. However, even these extensions are also not maximal (examples in Section 4). In this paper, we define a new ordering based on paths that is also total on ground terms (up to equivalence) and prove that it is maximal on non-ground terms. However, it is not compatible with RPO and compares some ground terms differently.

Computing the maximal extension of RPO can be done using a decision procedure to check if given two terms \( s \) and \( t \), there is a ground substitution \( \sigma \) that makes \( \sigma(s) \) bigger than \( \sigma(t) \) using RPO. Constraint solving procedures have been developed for this [1] [5]. This problem has also been proved NP-complete in [9]. In this paper, we develop a new method for computing the maximal extension of RPO using a bound on the depth of the substitution required.

The rest of this paper is organized as follows. In Section 2, we give the relevant definitions and background. In Section 3, we define a new ordering based on paths and prove it to be maximal. In Section 4, we show how to build a maximal extension of RPO using a new constraint solving procedure. We conclude in Section 5 with discussion of related work and suggestions for future work.