Confluence without Termination via Parallel Critical Pairs

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Abstract. We present a new criterion for confluence of (possibly) non-terminating left-linear term rewriting systems. The criterion is based on certain strong joinability properties of parallel critical pairs. We show how this criterion relates to other well-known results, consider some special cases and discuss some possible extensions.

1 Introduction and Overview

For abstract reduction (or abstract rewriting) systems (ARSs for short) it is well-known that, under termination, confluence is equivalent to local confluence, via Newman's Lemma. For proving confluence of non-terminating ARSs, however, one usually needs much stronger local confluence properties. A very interesting unifying framework, based on so-called decreasing diagrams, for localizing confluence proofs (even without termination) in the general setting of labelled abstract rewriting systems has recently been developed by van Oostrom ([8]).

For term rewriting systems (TRSs for short), which are ARSs with some additional structure, local confluence can be characterized by confluence of critical pairs as expressed by the well-known Critical Pair Lemma. Hence, for (finite) terminating TRSs, this critical pair test yields decidability of confluence.

For non-terminating TRSs, however, the situation is much more difficult again. Even the absence of critical pairs does not guarantee confluence, as there exist non-terminating, non-overlapping TRSs which are (locally confluent but) not confluent (cf. e.g. [3]). These counterexamples must necessarily be non-left-linear. In fact, TRSs which are left-linear and non-overlapping, i.e., orthogonal, are confluent (cf. e.g. [12]). This fundamentally important positive result has been considerably generalized by Huet ([3]) and further by Toyama ([13]) by allowing critical pairs, but imposing certain strong joinability properties on them (cf. Theorems 4, 6 and 8 below). Theorems 6 and 8 of Huet/Toyama are particularly interesting, since they do not require right-linearity. They are proved by showing strong confluence of parallel reduction, making essential use of the above mentioned particular joinability properties of (ordinary) critical pairs.

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2 Some interesting efforts for isolating the essence of the source of non-confluence in this case are described e.g. in [10], [15]. A decidable sufficient (syntactic) condition for confluence of non-left-linear, non-terminating TRSs is given in [15].
Our main new (and quite natural) idea now is that in order to ensure strong confluence of parallel reduction, one may also define and investigate the corresponding notion of parallel critical pairs. This concept indeed turns out to be very useful, since we are able to state and prove a new sufficient condition for strong confluence of parallel reduction (cf. Theorem 14) which is based on certain joinability properties for parallel critical pairs (cf. Definition 11).

Actually, the idea behind parallel critical pairs is not new. Implicitly, parallel critical pairs – or, more precisely, parallel critical peaks – are at the heart of so-called critical pair criteria for completion of terminating TRSs where certain (ordinary) critical pairs during completion can be ignored since they are redundant. Moreover, parallel critical pairs have also been used in another context for dealing with rewriting modulo some set of (non-orientable) equations ([11], [4]).

The rest of the paper is structured as follows. After introducing the necessary terminology, we present in Section 3 the known related results, give an example where none of the known confluence criteria applies, and motivate the introduction of parallel critical pairs. The main result of the paper, Theorem 14, is proved in Section 4. Its relation to the previous results, various illuminating examples and other related work are discussed in Section 5. Finally we conclude by discussing directions for further extending and generalizing our approach.

2 Preliminaries

We assume familiarity with the basic theory of abstract reduction systems as well as of the special case of term rewriting systems. For comprehensive surveys see e.g. [5], [2]. Furthermore we also use some basic facts about unification theory (cf. e.g. [1] for a recent survey).

2.1 Abstract Reduction Systems

An abstract reduction system (ARS) is a pair \( \mathcal{A} = (A, \rightarrow) \) consisting of a (base) set \( A \) and a binary relation \( \rightarrow \subseteq A \times A \) also called (abstract) reduction or (abstract) rewrite relation. We use the standard notations \( \rightarrow^+, \rightarrow^*, \rightarrow^= \) (or \( \rightarrow^{\leq 1} \)) for the transitive, transitive-reflexive and reflexive closure, respectively, of \( \rightarrow \). The notations for the inverse relations are obtained by ‘mirroring’, e.g., we use \( \leftarrow \) for the inverse of \( \rightarrow \). Relation composition is denoted by \( \circ \). Two elements \( a, b \in A \) are said to be joinable (denoted by \( a \parallel b \)) if there exists \( c \in A : a \rightarrow^* c \leftarrow^* b \). If \( a \rightarrow^* b \), we call \( b \) a reduct of \( a \).

Definition 1. (confluence properties)

Let \( \mathcal{A} = (A, \rightarrow) \) be an ARS. Then \( \mathcal{A} \) (or \( \rightarrow \))

- is confluent (CONF) if \( \rightarrow^\circ o \rightarrow^* \subseteq \rightarrow^* o \leftarrow^\circ \).
- is Church-Rosser (CR) if \( \leftarrow^* \subseteq \rightarrow^* o \leftarrow^* \).
- is locally confluent or weakly Church-Rosser (WCR) if \( \leftarrow o \rightarrow \subseteq \rightarrow^* o \leftarrow \).
- is strongly confluent (SCR) if \( \leftarrow o \rightarrow \subseteq \rightarrow^* o \leftarrow \).
- is subcommutative (WCR\( \leq 1 \)) if \( \leftarrow o \rightarrow \subseteq \rightarrow^= o \leftarrow \).

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