Quadratic Knapsack Relaxations Using Cutting Planes and Semidefinite Programming

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Abstract. We investigate dominance relations between basic semidefinite relaxations and classes of cuts. We show that simple semidefinite relaxations are tighter than corresponding linear relaxations even in case of linear cost functions. Numerical results are presented illustrating the quality of these relaxations.

1 Introduction

The quadratic knapsack problem is the easiest case of constrained 0/1 quadratic programming and is extremely difficult to solve by linear programming alone. Semidefinite programming is well known to provide powerful relaxations for quadratic 0/1 programming [7, 1, 4] and, as we intend to show, it is very useful for quadratic knapsack problems as well. We compare several possibilities for setting up initial relaxations and show that in the special case of linear cost functions some are even better than the canonical linear relaxation. We discuss possible strengthenings of these relaxations by polyhedral cutting plane approaches in theory and in practice. The main practical difficulty with semidefinite approaches is the high computational cost involved. These stem from the factorization of a completely dense symmetric positive definite matrix with dimension equal to the number of constraints. To keep the number of constraints small it is of major importance to understand the interaction and dominance relations between different classes of cuts. We give several theoretical results in this direction. Finally, we present computational results of this approach on practical data.

Let $N = \{1, \ldots, n\}$ be a set of items, $a \in \mathbb{N}^n$ a vector of weights, $b \in \mathbb{N}$ a capacity, and $C \in \mathbb{R}^{n \times n}$ a matrix of costs. The quadratic knapsack problem reads

$$\text{(QK)} \qquad \begin{array}{rl} \text{Maximize} & x^t C x \\ \text{subject to} & a^t x \leq b \\ & x \in \{0,1\}^n . \end{array}$$

We can interpret this problem in graph theoretic terms: Given the complete graph on n vertices with node weights a_i and profit c_{ii} for all $i = 1, \ldots, n$. Every edge ij in the complete graph is assigned an objective function coefficient

c_{ij}. Find a set of nodes S with sum of the node weights not greater than the threshold b that maximizes the profit $\sum_{i \in S} c_{ii} + \sum_{i,j \in S, i<j} 2c_{ij}$. As in the case of the linear knapsack problem the quadratic knapsack problem often appears as a subproblem to more complex optimization problems. Typical applications arise in VLSI- and compiler design [3, 6].

Our approach builds up on [4], which concentrates on the quadratic 0/1 programming aspects. Here, we investigate quadratic representations of a linear constraint, as suggested in [7, 1, 4] and discuss various aspects of knapsack specific inequalities.

The paper is structured as follows. Section 2 introduces several semidefinite relaxations obtained by different representations of the knapsack constraint and analyzes their strength. Section 3 surveys both well known and some new polyhedral concepts for generating knapsack specific cuts. In Section 4 we deal with the dominance relation between these cuts. In Section 5 implementational issues are discussed. We also present our numerical results.

2 Semidefinite Relaxation

(QK) is a constrained quadratic 0/1 programming problem. The usual approach for designing relaxations is to linearize the quadratic cost function by switching to "quadratic space". To this end we introduce variables y_{ij} for $i \leq j$ which are used to model the products $x_i x_j$. In the unconstrained case the convex hull of all feasible points in quadratic space is referred to as the boolean quadric polytope. The knapsack constraint cuts off part of this polytope. Although the convex hull of the restricted set of feasible integral points may differ substantially from the boolean quadric polytope it seems natural to start with a strong relaxation for the boolean quadric polytope and add knapsack specific inequalities on top.

Relaxation for the Boolean Quadric Polytope. As a relaxation for the boolean quadric polytope we use the semidefinite framework of [4] which is based on [7] and [1]. We model the dyadic product xx^t by a (symmetric) matrix variable Y. We denote the diagonal of this matrix by y. Using this notation the feasible set of matrices can be restricted to those satisfying $Y - yy^t \succeq 0$, i.e. $Y - yy^t$ must be positive semidefinite. This condition is equivalent to

$$\begin{pmatrix} Y & y \\ y^t & 1 \end{pmatrix} \succeq 0.$$

The diagonal elements y_i are obviously bounded by 0 and 1 and correspond to x_i. Looking at the determinant of a 3×3 principal minor containing the last row we get

$$y_i y_j - \sqrt{y_i y_j (1 + y_i y_j - y_i - y_j)} \leq y_{ij} \leq y_i y_j + \sqrt{y_i y_j (1 + y_i y_j - y_i - y_j)} \quad (1)$$

which yields an absolute lower bound of $-\frac{1}{8}$ for y_{ij}.