On the Complexity of Computational Problems Associated with Simple Stochastic Games
(Extended Abstract of COCOON'96)

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Abstract. We investigate simple stochastic games (SSGs): a kind of two-person games under uncertainty, the original model of which was introduced in [L.S. Shapley, Proc. Nat. Acad. Sci. U.S.A. 39 (1953) 1095-1100]. We consider the computational complexity of
1. deciding whether a given SSG is stopping (discounted) or not,
2. counting the number of all the optimal strategies of SSGs,
3. finding an optimal strategy against the player who takes random strategies.

1 Introduction

1.1 Investigated games and related problems

SSG: Simple stochastic games (SSGs) are a kind of stochastic game, the original model of which was introduced by Shapley [Sha53]. Formally, a simple stochastic game is defined over a pair of a directed graph \( G = (V, E) \) and a start vertex \( v_s \) with the following properties. The vertex set \( V \) is the union of disjoint sets \( V_{\text{max}} \), \( V_{\text{min}} \) and \( V_{\text{ave}} \), called respectively \( \text{max} \), \( \text{min} \) and \( \text{average} \) vertices, together with two special vertices, called the 0-goal and the 1-goal. The start vertex \( v_s \) is one vertex of \( V \). Each vertex of \( V \) has two outgoing edges, except the goal vertices which have no outgoing edges.

The game is played by two players, 0 and 1. Initially a token is placed on the start vertex, and at each step of the game the token is moved along edges of the graph, according to the following rules: At a min (resp. max) vertex, player 0 (resp. player 1) chooses an edge from that vertex and the token is moved along this edge. At an average vertex, the token is moved along an edge chosen randomly and uniformly from the vertex. The game ends when the token reaches a goal vertex. Player 1 wins if the token reaches the 1-goal; otherwise player 0 wins.

A strategy \( \tau \) of player 0 is a set of edges of \( E \), each edge has a min vertex at its left end, such that for each min vertex \( i \) there is exactly one edge \((i, j)\) in \( \tau \). Informally, if \((i, j) \in \tau \) then in a game where player 0 uses strategy \( \tau \), the token is always moved from vertex \( i \) to vertex \( j \). Similarly, a strategy \( \sigma \) of player 1 is a set of edges of \( E \), each edge has a max vertex at its left end, such that for each max vertex \( i \) there is exactly one edge \((i, j)\) in \( \sigma \).
Corresponding to strategy $\sigma$ is a graph $G_\sigma$, which is the subgraph of $G$ obtained by removing from each max vertex the outgoing edge that is not in the strategy $\sigma$. Similarly, corresponding to a pair of strategies $\sigma$ and $\tau$ is a graph $G_{\sigma,\tau}$ obtained from $G_\sigma$ by removing from each min vertex the outgoing edge that is not in $\tau$. In $G_{\sigma,\tau}$, every max and min vertex has one outgoing edge. If the number of max and min vertices is $k$, there are $2^k$ strategies.

Stopping problem: Some results on the complexity of SSGs are obtained under the assumption that games are the stopping type, namely, for all pairs of strategies, every vertex has a path to a goal vertex. Although there are no known efficient ways of checking a given game is stopping type or not, any game can be transformed into a stopping one in polynomial-time (using only logarithmic space) [Con92]. We note that the stopping SSGs are important class of SSGs and there are many results on stopping SSGs from a game-theoretic point of view [RCN73, Van77, Van78].

1.2 Our obtained results

This paper first gives a polynomial-time algorithm for the stopping problem for SSGs, and moreover shows that this problem is P-complete by reducing Alternative Graph Accessibility Problem [GHR89] into the problem.

Next this paper investigates the complexity of optimal strategies of simple stochastic games, and proposes a nondeterministic polynomial-time algorithm for counting the exact number of optimal strategies.

In an SSG, each player has one or more optimal strategies which ensure that the winning probability of the game is the best possible for one player, regardless of what the other player does. We call such a strategy an optimal strategy. In the game theory, finding optimal strategies is a fundamental problem, as is deciding which player has an advantage over the other.

Optimal strategies of SSGs are associated not only with a directed graph but also with a start vertex. However, in an SSG, there exist optimal strategies that are useful for any start vertex. We call such strategies universal optimal strategies. This paper focuses on the complexity of universal optimal strategies rather than optimal strategies associated with a start vertex. This is not a too serious restriction, because the complexity of finding a universal optimal strategy is shown to be the same (up to polynomial time) as of finding an optimal strategy that depends on the start vertex.

This paper shows that the set of all the universal optimal strategies of any stopping-type SSG has a polynomial-size certificate. By guessing such a certificate in a non-deterministic manner, we can efficiently enumerate the number of optimal strategies. Namely, the number of universal optimal strategies of any stopping SSG can be counted exactly in non-deterministic polynomial-time. Moreover, we have the consequence that, if the problem of computing a universal optimal strategy is solved in polynomial-time, then so is their counting problem.

If a given stopping SSG has only two types of vertices (average-max, average-min, or max-min), then we can find a universal optimal strategy in polynomial-time [Con92]. Thus we obtain the result that the number of universal optimal strategies of any stopping SSG with only (1) average and max vertices, or (2)