A New Efficiently Solvable Special Case of the
Three-Dimensional Axial Bottleneck
Assignment Problem *

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Abstract. Given an $n \times n \times n$ array $C = (c_{ijk})$ of real numbers, the three-
dimensional axial bottleneck assignment problem (3-BAP) is to find two
permutations $\phi$ and $\psi$ of $\{1, \ldots, n\}$ such that $\max_{i=1,\ldots,n} c_{i\phi(i)\psi(i)}$ is
minimized.

We first present two closely related conditions on the cost array $C$, the
wedge property and the weak wedge property, which guarantee that an
optimal solution of 3-BAP is obtained by setting $\phi$ and $\psi$ to the identity
permutation. In order to enlarge this class of efficiently solvable special
cases of the 3-BAP, we then propose an $O(n^3 \log n)$ time algorithm which,
given an $n \times n \times n$ array $C$, either finds three permutations $\rho$, $\sigma$ and $\tau$
such that the permuted array $C_{\rho,\sigma,\tau} = (c_{\rho(i)\sigma(j)\tau(k)})$ satisfies the wedge
property, or proves that no such permutations exist.

1 Introduction

Let an $n \times n \times n$ cost array $C = (c_{ijk})$ of real numbers be given and let $P_n$ denote
the set of permutations of $\{1, \ldots, n\}$. The three-dimensional axial bottleneck
assignment problem, or 3-BAP for short, is to find two permutations $\phi, \psi \in P_n$
such that the maximum cost entry given by

$$\max_{i=1,\ldots,n} c_{i\phi(i)\psi(i)}$$

is minimized. The 3-BAP is obtained from the well-known three-dimensional axial assignment problem, or 3-AP, by replacing the usual sum objective function
$\sum_{i=1}^n c_{i\phi(i)\psi(i)}$ by its bottleneck analogue (1). Most of the papers in the literature
on three-dimensional assignment problems deal with the sum case only. For
references on solution methods for the 3-AP, see e.g. Balas and Saltzman [1] or
Burkard and Rudolf [4]. For applications of three-dimensional axial assignment
problems, see Pierskalla [11, 12].

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By introducing 0-1 variables $x_{ijk}$ where $x_{ijk} = 1$ if and only if $\phi(i) = j$ and $\psi(i) = k$, the 3-BAP can be formulated as the following 0-1 integer program:

$$\min_{i,j,k=1,...,n} \max_{i,j,k=1,...,n} c_{ijk} x_{ijk}$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ijk} = 1 \quad \text{for all } k = 1, \ldots, n$$

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$$x_{ijk} \in \{0, 1\} \quad \text{for all } i, j, k = 1, \ldots, n.$$ 

It is easily seen that 3-AP resp. 3-BAP generalize the classical 2-dimensional assignment problem with sum resp. bottleneck objective. The term \textit{axial} in the name of the 3-AP and the 3-BAP is used to distinguish this type of three-dimensional assignment problem from a different one which is known as \textit{planar} three-dimensional assignment problem. In the planar case the set of feasible solutions is described by

$$\sum_{i=1}^{n} x_{ijk} = 1 \quad \text{for all } j, k = 1, \ldots, n$$

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$$x_{ijk} \in \{0, 1\} \quad \text{for all } i, j, k = 1, \ldots, n.$$ 

Hence a feasible solution in the planar case contains exactly $n^2$ variables with value 1, while in the axial case exactly $n$ variables have value 1. In the following we will only deal with axial three-dimensional assignments. From the point of view of special cases which is adopted in this paper, the axial version seems to be easier to attack due to the relatively small number of 1s in a feasible 0-1 solution.

\textbf{Previous results.} Both 3-AP and 3-BAP are known to be NP-hard even when the costs $c_{ijk}$ are restricted to be from \{0, 1\} (this follows from the NP-completeness of the three-dimensional matching problem proved in the seminal