Abstract. This paper studies the problem of coherence in category theory from a type-theoretic viewpoint. We first show how a Curry-Howard interpretation of a formal proof of normalization for monoids almost directly yields a coherence proof for monoidal categories. Then we formalize this coherence proof in intensional intuitionistic type theory and show how it relies on explicit reasoning about proof objects for intensional equality. This formalization has been checked in the proof assistant ALF.

1 Introduction

Mac Lane [18, pp.161–165] proved a coherence theorem for monoidal categories. A basic ingredient in his proof is the normalization of object expressions. But it is only one ingredient and several others are needed too.

Here we show that almost a whole proof of this coherence theorem is hidden in a Curry-Howard interpretation of a proof of normalization for monoids.

The second point of the paper is to contribute to the development of constructive category theory in the sense of Huet and Saibi [16], who implemented part of elementary category theory in the proof assistant Coq. Here we extend the scope of constructive category theory to the area of coherence theorems (cf. also [9]). We have formalized our proof in Martin-Löf type theory and implemented it in the proof assistant ALF. An interesting aspect of this formalization is that the problem of reasoning about explicit proofs of equality in the object language (arrows in a free monoidal category) reduces to reasoning about explicit proofs of equality in the metalanguage (proof objects for intensional equality I).

The paper is organized in the following way. In Section 2 we prove a normalization theorem for monoids. In Section 3 we introduce the notion of a monoidal category and prove coherence for it. In Section 4 we show how the proof can be formalized in intuitionistic type theory. Section 5 contains a few remarks about the implementation in ALF. Section 6 is about related work.

The ALF-implementation can be found on the web [11]. More discussion and a comparison with an implementation of the same proof in HOL can be found in Agerholm, Beylin, and Dybjer [2].
2 Normalization for Monoids

Let $M$ be the set of binary words with variables in the set $X$, that is, the least set such that

\[ e \in M \]
\[ x \in M \text{ for any } x \in X \]
\[ a \otimes b \in M \text{ for any } a, b \in M \]

Write $a \sim b$ if $a$ and $b$ are congruent with respect to associativity $(a \otimes (b \otimes c)) \sim (a \otimes b) \otimes c)$ and unit laws $(e \otimes a \sim a)$ and $a \otimes e \sim a)$. Hence $M/\sim$ is a free monoid generated by $X$.

Moreover, the subset $N$ of normal binary words is the least set such that
\[ e \in N \text{ and if } n \in N \text{ and } x \in X \text{ then } n \otimes x \in N. \]

We shall analyze the proof of the following “obvious” normalization theorem (see Hedberg [13]):

**Theorem 1.** There is a function (algorithm) $Nf : M \rightarrow N$, such that $a \sim b$ iff $Nf(a) = Nf(b)$.

A simple way to construct such a function is by using that $N^N$ together with function composition and the identity function forms a monoid. So let $Nf(a) = [a](e)$, where $[ ] : M \rightarrow N^N$ is defined by

\[
[e](n) = n
\]
\[
[x](n) = n \otimes x
\]
\[
[a \otimes b](n) = [b]([a](n))
\]

The theorem now follows from the following two lemmas:

**Lemma 2.** If $a \sim b$ then $[a] = [b]$ and $Nf(a) = Nf(b)$.

*Proof. By induction on the proof of $a \sim b$.***

**Lemma 3.** $a \sim Nf(a)$.

*Proof. $a \sim e \otimes a \sim [a](e) = Nf(a)$ using the following auxiliary lemma:*

**Lemma 4.** $n \otimes a \sim [a](n)$.

*Proof. By induction on $a$.

In the next section we shall see how a kind of Curry-Howard interpretation of this proof yields a proof of coherence for monoidal categories.