A New Characterization of $P_4$-connected Graphs

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Abstract. A graph is said to be $P_4$-connected if for every partition of its vertices into two nonempty disjoint sets, some $P_4$ in the graph contains vertices from both sets in the partition. A $P_4$-chain is a sequence of vertices such that every four consecutive ones induce a $P_4$. The main result of this work states that a graph is $P_4$-connected if and only if each pair of vertices is connected by a $P_4$-chain. Our proof relies, in part, on a linear-time algorithm that, given two distinct vertices, exhibits a $P_4$-chain connecting them. In addition to shedding new light on the structure of $P_4$-connected graphs, our result extends a previously known theorem about the $P_4$-structure of unbreakable graphs.

1 Introduction

Very recently, B. Jamison and S. Olariu [11] introduced the notion of $P_4$-connectedness. Specifically, a graph $G = (V, E)$ is $P_4$-connected if for every partition of $V$ into nonempty disjoint sets $V_1$ and $V_2$, some chordless path on four vertices and three edges (i.e. a $P_4$) contains vertices from both $V_1$ and $V_2$.

The concept of $P_4$-connectedness leads to a structure theorem for arbitrary graphs in terms of $P_4$-connected components and suggests, in a quite natural way, a tree representation unique up to isomorphism. The leaves of the resulting tree are the $P_4$-connected components and the weak vertices, that is, vertices belonging to no $P_4$-connected component.

This structure theorem and the corresponding tree representation, on the one hand, provide tools for the study of graphs with a simple $P_4$-structure, such as $P_4$-reducible [8], $P_4$-extendible [9], $P_4$-sparse [10] and, more generally, $(q,t)$ graphs [1]. On the other hand, and more importantly, the structure theorem lays the foundation of the so-called homogeneous decomposition of graphs [11], a decomposition which can be seen as an extension of the well known modular decomposition (also called substitution decomposition [13]). In this context, it is of particular interest to investigate graphs which are prime with respect to the homogeneous decomposition. It is an immediate consequence of the structure theorem that those graphs are $P_4$-connected.

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Further strong motivation for the study of $P_4$-connected graphs is provided by their relationship to unbreakable graphs. These graphs are known to play a central role in attempts to reveal the structure of minimal counterexamples to the famous, and yet unresolved, Strong Perfect Graph Conjecture.

The concept of $P_4$-connectedness obviously generalizes the usual connectedness of graphs, since a graph $G = (V, E)$ is connected, in the usual sense, if for every partition of $V$ into nonempty disjoint sets $V_1$ and $V_2$ there exists an edge with one endpoint in $V_1$ and the other in $V_2$. A more common characterization states that each pair of different vertices is connected by a path. The main contribution of this paper is to point out a similar characterization of $P_4$-connected graphs in terms of so-called $P_4$-chains.

The remainder of this work is organized as follows. Section 2 establishes terminology and summarizes previous art. Section 3 reviews a number of results about unbreakable graphs and shows that unbreakable graphs are $P_4$-connected. In Section 4 we introduce the concept of a $P_4$-chain and present our main result, a characterization of $P_4$-connected graphs in terms of $P_4$-chains. As a corollary, we obtain an extension of a result of Chvátal [5] on unbreakable graphs. In Section 5, we present a linear-time algorithm that, given a pair of vertices in a $P_4$-connected graph, constructs a $P_4$-chain connecting them.

2 Basics and Terminology

All graphs in this paper are finite, with no loops nor multiple edges. In addition to standard graph-theoretical terminology, compatible with [2], we need new terms that we are about to define.

Let $G = (V, E)$ be a graph with vertex-set $V$ and edge-set $E$. For a vertex $v$ of $G$, $N(v)$ denotes the set of all vertices adjacent to $v$. If $U \subseteq V$ then $G(U)$ stands for the graph induced by $U$. Occasionally, to simplify the exposition, we shall blur the distinction between sets of vertices and the subgraphs they induce, using the same notation for both. The complement of $G$ is denoted by $\overline{G}$. A clique is a set of pairwise adjacent vertices, a stable set is a set of pairwise nonadjacent vertices. $G$ is termed a split graph if its vertices can be partitioned into a clique and a stable set.

We say that two sets $X$ and $Y$ of vertices of $G$ are nonadjacent if no edge has one endpoint in $X$ and the other in $Y$. $X$ and $Y$ are totally adjacent if every vertex in $X$ is adjacent to all vertices in $Y$. Finally, sets $X$ and $Y$ that are neither nonadjacent nor totally adjacent are termed partially adjacent.

A vertex $v$ is said to distinguish a set $U$ of vertices if $v$ is partially adjacent to $U$. A subset $Z$ of $V$ with $1 < |Z| < |V|$ is termed a homogeneous set if no vertex outside $Z$ distinguishes $Z$, i.e. each vertex outside $Z$ is nonadjacent or totally adjacent to $Z$. The graph obtained from $G$ by shrinking every maximal homogeneous set to one single vertex is called the characteristic graph of $G$.

As usual, we let $P_k$ stand for the chordless path with $k$ vertices and $k - 1$ edges. In a $P_4$ with vertices $u, v, w, x$ and edges $uv, vw, wx$, vertices $v$ and $w$ are referred to as midpoints whereas $u$ and $x$ are called endpoints.