Parallel Iterative Solvers for Banded Linear Systems

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Abstract. A parallel implementation of the SOR iterative method is presented for the solution of block banded linear systems. The algorithm is based on the block reordering of the coefficient matrix used by the domain decomposition methods. It is proved that the obtained iteration matrix maintains the same spectral properties of the corresponding sequential method and also the same optimal parameter of relaxation. The parallel SOR algorithm is then applied to the solution of linear systems arising from the discretization of elliptic partial differential equations in order to obtain an interesting comparison with the coloring schemes.

Key words and phrases. Parallel algorithms, iterative solvers, SOR iteration.

1 Introduction

Numerical linear algebra is the kernel of most of the existing algorithms for the solution of problems arising from physics, chemistry, and engineering. The use of parallel computers has reduced the execution time and has allowed the solution of quite difficult problems such as, for example, those derived from the meteorology.

In this paper we are interested in iterative methods for the parallel solution of large banded linear systems. Banded systems (in particular tridiagonal systems) arise from the discretization of several PDEs and ODEs [2, 4, 7]. Their parallel solution by means of the classical iterations (Gauss-Seidel, SOR) has received particular attention especially when the coefficient matrices arise from the discretization of elliptic partial differential equations on rectangular domains. In particular, most of the approaches are based on the SOR iteration applied to different orderings, called coloring, of the grid nodes (see [1, 2]).

The easiest and most famous ordering is called red/black, since it corresponds to perform a red/black (or odd/even) permutation to the problem given by the natural rowwise ordering. An important property of this ordering is that its iteration matrix has the same eigenvalues of that corresponding to the natural rowwise. Therefore, the algorithm associated to the red/black has the same asymptotic convergence rate of the sequential SOR.
In this paper we analyze a class of parallel iteration schemes which preserves the convergence properties of the natural rowwise scheme. The algorithms are based on the partitioning of the coefficient matrix among the processors used by the domain decomposition methods [3, 7]. They are obtained by applying classical iterative methods to a permuted problem, where the permutation matrix is chosen in order to emphasize parallel computations.

In Section 2 we present the parallel block and point SOR algorithms applied to block banded matrices, without considering the internal structure of each block. In Section 3 we analyze the parallel solution of elliptic boundary value problems by exploiting the internal band structure of each block. In this way some new iteration schemes, similar to the classical multicoloring schemes, are derived.

2 The parallel SOR method

Let us consider the solution of the linear system

$$Ax = f,$$

where the coefficient matrix $A$ is block banded

$$A = \begin{pmatrix} A_1 & C_{1,1} & \cdots & C_{r,1} \\ B_{1,2} & A_2 & C_{1,2} & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ B_{t,s+1} & \cdots & \cdots & A_{n-1} & C_{1,n-1} \\ B_{t,n} & \cdots & B_{1,n} & A_n \end{pmatrix},$$

and the blocks $A_i$, $B_{ij}$ and $C_{ij}$ are $m \times m$ matrices.

Let $p \leq \min(n/(2r), n/s)$ be an integer that exactly divides $n$, consider the following partitioning of $A$:

$$A = \begin{pmatrix} M_1 & -U_1' \\ -L_2' & M_2 \end{pmatrix},$$

where $M_i = D_i - U_i - L_i$ are $(n/p) \times (n/p)$ block banded matrices with $D_i$ as their main block diagonal and $-L_i$ and $-U_i$ as their lower and upper block triangular part; $L_i'$ and $U_i'$ are block matrices respectively with $s$ off-diagonals in the upper right corner and $r$ off-diagonals in the lower left corner.

Moreover, let