The Wadge-Wagner Hierarchy of \( \omega \)-Rational Sets

Olivier Carton and Dominique Perrin

Institut Gaspard Monge
Université de Marne-la-Vallée
93166 Noisy le Grand
France

Abstract. We present a unified treatment of the hierarchy defined by Klaus Wagner for \( \omega \)-rational sets and also introduced in the more general framework of descriptive set theory by William W. Wadge. We show that this hierarchy can be defined by syntactic invariants, using the concept of an \( \omega \)-semigroup.

1 Introduction

The idea of a Muller automaton was introduced by David Muller as a variant of usual finite automata, well suited for the recognition of infinite sequences. It was later proved by McNaughton that any recognizable set of \( \omega \)-words can be recognized by a deterministic Muller automaton.

Klaus Wagner has introduced in 1979 [22] two concepts defined on Muller automata: chains and superchains. Together with an operation on automata called derivation, he has proved that the maximal lengths of chains and superchains (and the ones obtained on the derived automata) are enough to characterize the classes of recognizable \( \omega \)-sets up to the inverse image under a continuous function. This classification has also been investigated independently by W. Wadge. He has studied the reduction by a continuous function in abstract topological spaces, as a refinement of the classical Borel hierarchy. His results are based on a particular class of games, now called Wadge games. His classification itself is known as the Wadge hierarchy [10]. The connections between both theories were first discovered by Pierre Simonnet [19]. The Wagner hierarchy has been partially rediscovered several times [2, 9]. The interest in the classification of \( \omega \)-rational sets was revived by the studies concerning the logic of distributed processing [15].

Since then Thomas Wilke [24] has shown how one could use, in the case of infinite words, algebraic methods allowing to replace finite automata by finite semigroups. This has lead to the notion of an \( \omega \)-semigroup introduced in [17]. This approach has the advantage to make easier the definition of a variety along the line of Eilenberg’s theory.

Another direction was investigated by Jean-Eric Pin in [18]. He has shown that the notion of ordered semigroup could be used to define families of recognizable sets that are not closed under complementation. This is especially interesting in the case of infinite words since very natural families like the open sets are not closed under complementation.
We would like to show here how Klaus Wagner's ideas fit into the present framework using \( \omega \)-semigroups. In particular, we shall see that the definition of chains and superchains can be formulated in \( \omega \)-semigroups, providing a clear explanation of the fact that they do not depend on the particular automaton used to recognize a given set but on the set itself. We shall show how the classes of the Wagner hierarchy are defined in topological terms. We will also investigate the link between Wagner's notions and that of ordered semigroups.

The work presented here is based on results obtained, in great part, in the first author doctoral thesis [4]. Part of it was presented at a conference held in Porto [6]. Those concerning the equivalence of the various definitions of chains and superchains will appear soon in [7]. The ones concerning the hierarchy itself will be published in a second paper [5].

2 Preliminaries

We assume a familiarity with the basic concepts of \( \omega \)-rational sets and automata. For an introduction, we refer the reader to [21] or [16]. A word about notation. The alphabet is usually denoted by the symbol \( A \). The set \( A^* \) (resp. \( A^+ \)) is the set of finite words (resp. nonempty finite words) on the alphabet \( A \). The set of (one-sided) infinite words on \( A \) is denoted by \( A^\omega \). We consider \( A^\omega \) as a topological space with the usual Cantor topology.

We shall deal often with classes of sets. Since the sets considered are subsets of the topological space \( A^\omega \), a class of sets is really a mapping assigning to each alphabet \( A \) a set of subsets of \( A^\omega \). The dual class of a class \( F \) is formed of the complements (within each \( A^\omega \)) of the sets in \( F \). It is denoted by \( \bar{F} \). We say that \( F \) is ambiguous if \( F = \bar{F} \).

We shall use ordinals to index classes of sets. The symbol \( \omega \) will thus be used in two ways, either to denote an ordinal in expressions like \( \omega + 1 \) or to denote an \( \omega \)-rational set like \( (a^* b)^\omega \). We hope that it will not bring confusion.

We now recall the definition of \( \omega \)-semigroups and Wilke algebras. For a more detailed presentation, we refer the reader to [17]. We assume some familiarity with the basic notions of semigroup theory. We use the notation of [8] for all undefined notions in semigroup theory. We use the traditional notation \( S^1 \) to denote the semigroup obtained by adding an new neutral element 1 to \( S \).

An \( \omega \)-semigroup is a pair \( S = (S_+, S_\omega) \) where \( S_+ \) is a semigroup and \( S_\omega \) is a set with two operations in addition to the semigroup operation of \( S_+ \): A left action of \( S_+ \) on \( S_\omega \):

\[
(s, u) \mapsto s.u
\]

and an infinite product

\[
\pi : S_+ \times S_+ \times S_+ \times \ldots \rightarrow S_\omega
\]

These operations must satisfy the following axioms:

1. The action of \( S_+ \) on \( S_\omega \) is associative: for \( s, t \in S_+ \) and \( u \in S_\omega \)

\[
s.(t.u) = (st).u
\]