Computability on the Probability Measures on the Borel Sets of the Unit Interval

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Abstract. While computability theory on many countable sets is well established and for computability on the real numbers several (mutually non-equivalent) definitions are applied, for most other uncountable sets, in particular for measures, no generally accepted computability concepts at all have been available until now. In this contribution we introduce computability on the set $\mathcal{M}$ of probability measures on the Borel subsets of the unit interval $[0; 1]$. Its main purpose is to demonstrate that this concept of computability is not merely an ad hoc definition but has very natural properties. Although the definitions and many results can of course be transferred to more general spaces of measures, we restrict our attention to $\mathcal{M}$ in order to keep the technical details simple and concentrate on the central ideas. In particular, we show that simple obvious requirements exclude a number of similar definitions, that the definition leads to the expected computability results, that there are other natural definitions inducing the same computability theory and that the theory is embedded smoothly into classical measure theory. As background we consider TTE, Type 2 Theory of Effectivity [KW84, KW85], which provides a frame for very realistic computability definitions. In this approach, computability is defined on finite and infinite sequences of symbols explicitly by Turing machines and on other sets by means of notations and representations. Canonical representations are derived from information structures [Wei97]. We introduce a standard representation $\delta_m : \Sigma^* \longrightarrow \mathcal{M}$ via some natural information structure defined by a subbase $\sigma$ (the atomic properties) of some topology $\tau$ on $\mathcal{M}$ and a standard notation of $\sigma$. While several modifications of $\delta_m$ suggesting themselves at first glance, violate simple and obvious requirements, $\delta_m$ has several very natural properties and hence should induce an important computability theory. Many interesting functions on measures turn out to be computable, in particular linear combination, integration of continuous functions and any transformation defined by a computable iterated function system with probabilities. Some other natural representations of $\mathcal{M}$ are introduced, among them a Cauchy representation associated with the Hutchinson metric, and proved to be equivalent to $\delta_m$. As a corollary, the final topology $\tau$ of $\delta_m$ is the well known weak topology on $\mathcal{M}$. 
1 Introduction

Measure and integration is a central branch of mathematics pervading almost all parts of abstract analysis. Several authors have already considered questions of effectivity, constructivity, computability or computational complexity in measure or integration theory. Kushner [Kus85] studies computability and Ko [Ko9t] computational complexity of integration. Bishop and Bridges [BB85] present constructive measure theory extensively. Although they do not consider computability, certainly many of their concepts and results have computational counterparts. Edalat gives a domain theoretic approach to effective integration [Eda95, Eda96]. He also does not consider computability, but it should be possible to extend his topological approach by computability concepts. Traub et al. [TWW88] investigate the computational complexity of numerical algorithms for integration in the real number model of computation. However, this model is unrealistic in many situations and therefore not generally accepted. A systematic study of computability in integration and measure theory does not yet exist.

In this paper we introduce a very natural and realistic computability theory on probability measures. We achieve this by extending TTE, Type 2 Theory of Effectivity, to measure theory. TTE has been introduced by Kreitz and Weihrauch [KW84, KW85] as a general framework for studying effectivity, i.e. continuity, computability and computational complexity, in Analysis. For details the reader is referred to the introduction [Wei95] and a recent short survey [Wei97] containing most of the notations we shall use in this paper. More details can be found in [KW85, Wei87]. Since this paper is a first attempt, we consider only the space of probability measures on the Borel subsets of the real unit interval.

By \( f : A \rightarrow B \) we denote a partial function, i.e. a function from a subset of \( A \) to \( B \). Throughout this paper let \( \Sigma \) be a sufficiently large finite alphabet. Let \( \Sigma^* \) be the set of finite and \( \Sigma^\omega = \{ p \mid p : \omega \rightarrow \Sigma \} \) the set of "ega-words" over \( \Sigma \). On \( \Sigma^* \) we consider the discrete topology and on \( \Sigma^\omega \) the cantor topology defined by the basis \( \{ w \Sigma^\omega \mid w \in \Sigma^* \} \). For \( Y_0, Y_1, \ldots, Y_k \in \{ \Sigma^*, \Sigma^\omega \} \), a function \( f : Y_1 \times \ldots \times Y_k \rightarrow Y_0 \) is called computable, iff it is computed by a Turing machine with a one-way output tape. Every computable function is continuous. The basic idea of TTE is to use finite or infinite sequences as names of "abstract" objects. As naming systems we consider notations, i.e. surjections \( \nu : \Sigma^* \rightarrow \mathbb{N} \), and representations, i.e. surjections \( \delta : \Sigma^\omega \rightarrow \mathbb{M} \). Continuity and computability concepts are transferred from \( \Sigma^* \) and \( \Sigma^\omega \) via notations and representations, respectively, to the named sets straightforwardly, see [KW85, Wei87, Wei95, Wei97]. Mainly notations or representations which are compatible with some relevant structure on the set under consideration are of practical interest. We do not discuss this for notations (see [RW80, Wei87] and Appendix C in [Wei95]), but we will introduce "effective" notations explicitly whenever necessary. In particular, for the rational numbers let \( \nu_\mathbb{Q} : \Sigma^* \rightarrow \mathbb{Q} \) be the standard representation via fractions of integers in binary notation. We shall abbreviate \( \nu_\mathbb{Q}(w) \) by \( \bar{w} \). Standard notations of the natural numbers, pairs of rational numbers etc. will be used without further definitions. For uncountable sets \( \mathbb{M} \) we shall consider mainly representations derived from "information