Distance Approximating Trees for Chordal and Dually Chordal Graphs*
(Extended Abstract)

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Abstract. In this note we show that, for each chordal graph $G$, there is a tree $T$ such that $T$ is a spanning tree of the square $G^2$ of $G$ and, for every two vertices, the distance between them in $T$ is not larger than the distance in $G$ plus two. Moreover, we prove that, if $G$ is a strongly chordal graph or even a dually chordal graph, then there exists a spanning tree $T$ of $G$ which is an additive 3-spanner as well as a multiplicative 4-spanner of $G$. In all cases the tree $T$ can be computed in linear time.

1 Introduction

Many combinatorial and algorithmic problems concern the distance $d_G$ on the vertices of a possibly weighted graph $G = (V, E)$. Approximating $d_G$ by a simpler distance (in particular, by tree-distance) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis (see [1, 2, 4, 8, 10, 20, 25, 26, 28, 30]). The goal is, for a given graph $G$, to find a sparse graph $H = (V, E')$ with the same vertex-set, such that the distance $d_H(u, v)$ in $H$ between two vertices $u, v \in V$ is reasonably close to the corresponding distance $d_G(u, v)$ in the original graph $G$. There are several ways to measure the quality of this approximation, two of them leading to the notion of a spanner. For $t \geq 1$ a spanning subgraph $H$ of $G$ is called a \textit{multiplicative $t$-spanner} of $G$ [26, 10, 25] if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$. If $r \geq 0$ and $d_H(u, v) \leq d_G(u, v) + r$ for all $u, v \in V$, then $H$ is called an \textit{additive $r$-spanner} [20].

For many applications (e.g. in numerical taxonomy or in phylogeny reconstruction) the condition that $H$ must be a spanning subgraph of $G$ can be dropped (see [2, 28, 30]). In this case there is a striking way to measure how sharp $d_H$ approximates $d_G$, based on the notion of a pseudoisometry between two metric spaces. This idea is borrowed from the geometry of hyperbolic groups

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For graphs and finite metric spaces a related notion of a near-isometry has been already used by Linial et al [21]. For our purposes we present a simplified version of this notion (the interested reader can consult [13, pp.71–72] and [16] for the general definition and related material).

Let $t \geq 1$ and $r \geq 0$ be real numbers. Two graphs $G = (V, E)$ and $H = (V, E')$ are called $(t, r)$-pseudoisometric if

$$d_H(u, v) \leq t \cdot d_G(u, v) + r \quad \text{and} \quad d_G(u, v) \leq t \cdot d_H(u, v) + r$$

for all $u, v \in V$. In this case we will say that $H$ is a distance $(t, r)$-approximating graph for $G$ (and conversely, $G$ will be a distance $(t, r)$-approximating graph for $H$). The graphs $G$ and $H$ are $(t, 0)$-pseudoisometric iff

$$\frac{1}{t} \cdot d_G(u, v) \leq d_H(u, v) \leq t \cdot d_G(u, v)$$

for $u, v \in V$. If, in addition, $H$ is a spanning subgraph of $G$, then we obtain the notion of the multiplicative $t$-spanner. Clearly, $G$ and $H$ are $(1, r)$-pseudoisometric iff $|d_G(u, v) - d_H(u, v)| \leq r$ for $u, v \in V$. Again, if $H$ is a spanning subgraph of $G$, this is the usual notion of the additive $r$-spanner.

Recently Cai and Corneil [8] have considered multiplicative tree spanners in graphs. They showed that for a given graph $G$ and integer $t$, the problem to decide whether $G$ has a tree $t$-spanner is NP-complete for $t \geq 4$ and is linearly solvable for $t = 1, 2$. The status of the case $t = 3$ is still open. Tree $3$-spanners exist for interval and permutation graphs and they can be found in linear time [22]. Similar results are known for the additive tree $r$-spanner problem. [27] proposes a simple approach to construct additive tree $2$-spanners in interval and distance-hereditary graphs and such $4$-spanners in cocomparability graphs. Both papers [8, 27] ask which important graph classes have tree $t$-spanners and $r$-spanners with small $t$ and $r$. As it is mentioned in [27], McKee showed that for every fixed integer $t$ there is a chordal graph without tree $t$-spanners (additive as well as multiplicative). Nevertheless, from the metric point of view chordal graphs look like trees. In this note we prove that for every chordal graph $G$ there exists a tree $T$ (actually, $T$ is a spanning tree of the square $G^2$) such that

$$d_T(u, v) \leq 3 \cdot d_G(u, v) \quad \text{and} \quad d_T(u, v) \leq d_G(u, v) + 2$$

for all vertices $u, v$ of $G$. In other words, $T$ is a $(3, 0)$- and $(1, 2)$-approximating tree for $G$. Moreover, if $G$ is a strongly chordal graph then there exists a spanning tree $T$ of $G$ which is an additive 3-spanner and a multiplicative 4-spanner. Thus, this answers the question whether strongly chordal graphs have tree $t$-spanners with small $t$, posed in [27]. Furthermore we show that the method elaborated for strongly chordal graphs works for a more general graph class, for the dually chordal graphs. In all cases the tree $T$ can be computed in linear time.