Optimal Periodic Control with Environmental Application

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Abstract. We consider the problem of designing an optimal control for linear discrete-time systems assuming that total cost for the control efforts is limited and cost function is periodic in behaviour. (seasonal, for example) This Model was developed as a result of analysis of real data of the project "Modelling River Murray Estuary" from the Environmental Modelling Research Group, the University of South Australia.

Nonlinear control systems are of a great significance in the field of control engineering since most practical dynamic systems are nonlinear. Using arbitrary control strategy as an initial we can compute coefficients in nonlinear system as a function of corresponding output variables. (previous output variables) As a result we shall transform nonlinear system into linear system with known optimal solution. (new output variables) Repeating this procedure again and again we shall generate sequence of control strategies. Optimal control strategy for given nonlinear system may be obtained as a limit of this sequence. This fact has been demonstrated by the particular example relating to the above environmental research Project.

1 Linear Model

Consider the discrete-time system described by the linear difference equation

\[ x_{k+1} = A(k)x_k + B(k)u_k \]  

where \( x_k \geq 0 \) is the system state; \( u_k \geq 0 \) is the control; and the coefficients \( 0 < A(k) < 1, B(k) > 0 \) are known, \( k=1, 2, \ldots \)

Suppose that cost function for the control efforts and coefficients in (1) are periodic in behaviour:

\[ q(k+n) = q(k), A(k+n) = A(k), B(k+n) = B(k) \]

where \( n \) is a positive integer and total cost for one period is limited

\[ Q = \sum_{k=1}^{n} q(k)u_k < \infty. \]  

According to the above assumptions we shall suppose that \( u_k = u_{k+n}, x_k = x_{k+n} \) for any \( k=1,2,\ldots \) and
\[
\begin{align*}
\sum_{k=1}^{n-1} B(k)u_k \prod_{j=k+1}^n A(j) + B(n)u_n = x_1 = \frac{1 - \prod_{j=1}^n A(j)}{1 - \prod_{j=k+1}^n A(j)} \quad (3)
\end{align*}
\]

**Remark.** There is a one-to-one correspondence between control and state. The task is to find optimal control strategy (or optimal system state) in order to minimise the following loss function:

\[
\Psi = \sum_{k=1}^{n} \frac{1}{(x_k + a)^\lambda}, \quad a \in R, \lambda > 0. \quad (4)
\]

Using (1) we can rewrite (2):

\[
Q = \sum_{k=1}^{n} \alpha_k x_k \quad (5)
\]

where \( \alpha_1 = \frac{q(n) - A(1)q(1)}{B(n)} \), \( \alpha_k = \frac{q(k-1) - A(k)q(k)}{B(k-1) - B(k)}, k = 2..n. \)

**Remark.** Suppose that the coefficients \( A(k) \) are small enough and \( \alpha_k > 0, k = 1..n. \)

**Theorem.** The optimal system state for the above task is unique and is specified by the following formulas:

\[
x_k = \frac{Q + aI}{I_\lambda \alpha_k^{\lambda+1}} - a, \quad I = \sum_{j=1}^{n} \alpha_j, \quad \lambda = \sum_{j=1}^{n} \alpha_j^{\lambda+1}; \quad \min_{u_k, k=1..n} \Psi = \frac{I_\lambda^{\lambda+1}}{(Q + aI)^\lambda}. \quad (6)
\]

**Proof.** Let us apply (5) to (4):

\[
\Psi = \sum_{k=1}^{n-1} \frac{1}{(x_k + a)^\lambda} + \frac{\alpha^\lambda}{(Q - \sum_{j=1}^{n-1} \alpha_j x_j + a\alpha_n)^\lambda}.
\]

Differentiating above with respect to \( x_k, k=1..n-1 \), we obtain the system of \( n-1 \) equations:

\[
Q - \sum_{j=1}^{n-1} \alpha_j x_j + \alpha_n a = \alpha_k^{\lambda+1} \alpha_n^{\lambda+1} (x_k + a)
\]

with solution...