The Combined Shepard-Lidstone Bivariate Operator

Teodora Cătinaş

Abstract. We extend the Shepard operator by combining it with the Lidstone bivariate operator. We study this combined operator and give some error bounds.

1. Preliminaries
1.1. The Shepard bivariate operator
Recall first some results regarding the Shepard operator for the bivariate case [7], [17]. Let \( f \) be a real-valued function defined on \( X \subset \mathbb{R}^2 \), \( (x_i, y_i) \in X \), \( i = 0, \ldots, N \) some distinct points and \( r_i (x, y) \), the distances between a given point \( (x, y) \in X \) and the points \( (x_i, y_i) \), \( i = 0, 1, \ldots, N \).

The Shepard interpolation operator is defined by

\[
(S f)(x, y) = \sum_{i=0}^{N} A_i (x, y) f (x_i, y_i),
\]

where

\[
A_i (x, y) = \frac{\prod_{j=0}^{N} r_{ij}^\mu (x, y)}{\sum_{k=0}^{N} \prod_{j=0}^{N} r_{kj}^\mu (x, y)},
\]

with \( \mu \in \mathbb{R}_+ \).

It follows that

\[
\sum_{i=0}^{N} A_i (x, y) = 1.
\]

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Because of its small degree of exactness we are interested in extending the Shepard operator $S$ by combining it with some other operators. Let $\Lambda := \{\lambda_i: i = 0, \ldots, N\}$ be a set of functionals and $P$ the corresponding interpolation operator. We consider the subsets $\Lambda_i \subset \Lambda$, $i = 0, \ldots, N$ such that $\bigcup_{i=0}^{N} \Lambda_i = \Lambda$ and $\Lambda_i \cap \Lambda_j \neq \emptyset$, excepting the case $\Lambda_i = \{\lambda_i\}$, $i = 0, \ldots, N$, when $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$. We associate the interpolation operator $P_i$ to each subset $\Lambda_i$, for $i = 0, \ldots, N$.

The combined operator of $S$ and $P_i$, denoted by $S_P$, is defined by

$$(S_P f)(x, y) = \sum_{i=0}^{N} A_i(x, y) (P_i f)(x, y).$$

Remark 1.1. [16] If $P_i$, $i = 0, \ldots, N$, are linear and positive operators then $S_P$ is a linear and positive operator.

Remark 1.2. [16] Let $P_i$, $i = 0, \ldots, N$, be some arbitrary linear operators. If $\text{dex}(P_i) = r_i$, $i = 0, \ldots, N$, then $\text{dex}(S_P) = \min\{r_0, \ldots, r_N\}$.

1.2. Two variable piecewise Lidstone interpolation
We recall first some results from [1] and [2]. Consider $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ and let $\Delta : a = x_0 < x_1 < \ldots < x_{N+1} = b$ and $\Delta' : c = y_0 < y_1 < \ldots < y_{M+1} = d$ denote uniform partitions of the intervals $[a, b]$ and $[c, d]$ with stepsizes $h = (b-a)/(N+1)$ and $l = (d-c)/(M+1)$, respectively. Denote by $\rho = \Delta \times \Delta'$ the resulting rectangular partition of $[a, b] \times [c, d]$. For the univariate function $f$ and the bivariate function $g$ and each positive integer $r$ we denote by $D^r f = d^r f / dx^r$, $D'_x g = \partial^r g / \partial x^r$ and $D'_y g = \partial^r g / \partial y^r$.

Definition 1.3. [2] For each positive integer $r$ and $p$, $1 \leq p \leq \infty$, let $PC^{r,p}[a, b]$ be the set of all real-valued functions $f$ such that:

(i) $f$ is $(r - 1)$ times continuously differentiable on $[a, b]$;
(ii) there exist $s_i$, $0 \leq i \leq L + 1$ with $a = s_0 < \ldots < s_{L+1} = b$, such that on each subinterval $(s_i, s_{i+1})$, $0 \leq i \leq L$, $D^{r-1} f$ is continuously differentiable;
(iii) the $L_p$-norm of $D^r f$ is finite, i.e.,

$$\|D^r f\|_p = \left( \sum_{i=0}^{L} \int_{s_i}^{s_{i+1}} |D^r f(x)|^p \, dx \right)^{1/p} < \infty.$$

For the case $p = \infty$ it reduces to

$$\|D^r f\|_\infty = \max_{0 \leq i \leq L} \sup_{x \in (s_i, s_{i+1})} |D^r f(x)| < \infty.$$

Definition 1.4. [2] For each positive integer $r$ and $p$, $1 \leq p \leq \infty$, let $PC^{r,p}([a, b] \times [c, d])$ be the set of all real-valued functions $f$ such that:

(i) $f$ is $(r - 1)$ times continuously differentiable on $[a, b] \times [c, d]$, i.e., $D^\mu_x D^\nu_y f$, $0 \leq \mu + \nu \leq r - 1$, exist and are continuous on $[a, b] \times [c, d]$;