Minimal Models for $\mathcal{N}_\kappa^\infty$-functions

Aad Dijksma, Annemarie Luger and Yuri Shondin

Abstract. We present explicit realizations in terms of self-adjoint operators and linear relations for a non-zero scalar generalized Nevanlinna function $N(z)$ and the function $\hat{N}(z) = -1/N(z)$ under the assumption that $\hat{N}(z)$ has exactly one generalized pole which is not of positive type namely at $z = \infty$. The key tool we use to obtain these models is reproducing kernel Pontryagin spaces.

Mathematics Subject Classification (2000). Primary 47B25, 47B50, 47B32; Secondary 47A06.

Keywords. Generalized Nevanlinna function, generalized pole, realization, model, reproducing kernel spaces, Pontryagin spaces, self-adjoint operator, symmetric operator, linear relation, block operator matrix.

1. Introduction

An $n \times n$ matrix function $N$ is called a generalized Nevanlinna function with $\kappa$ negative squares if (i) it is defined and meromorphic on $\mathbb{C} \setminus \mathbb{R}$, (ii) it satisfies $N(z) = N(z^*)^*$ for all $z \in \mathcal{D}(N)$, the domain of holomorphy of $N$, and (iii) the kernel

$$K_N(\zeta, z) = \frac{N(\zeta) - N(z)^*}{\zeta - z^*}, \quad \zeta, z \in \mathcal{D}(N),$$

has $\kappa$ negative squares. Here the expression on the right-hand side for $\zeta = z^*$ is to be understood as $N'(\zeta)$. If $\kappa = 0$, the function $N$ is called a Nevanlinna function;

The authors gratefully acknowledge support from the “Fond zur Förderung der wissenschaftlichen Forschung” (FWF, Austria, grant number P15540-N05), the Netherlands Organization for Scientific Research NWO (grant NWO 047-008-008), and the Research Training Network HPRN-CT-2000-00116 of the European Union.
in this case \( N \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R} \), satisfies \( N(z) = (N(z^*)^* \) there and the kernel condition is equivalent to the condition
\[
\frac{\text{Im} N(z)}{\text{Im} z} \geq 0, \quad \text{Im} z \neq 0.
\]
The class of \( n \times n \) matrix functions with \( \kappa \) negative squares is denoted by \( \mathcal{N}_{\kappa}^{n \times n} \) and by \( \mathcal{N}_{\kappa} \) when the functions are scalar.

A realization for a function \( N \in \mathcal{N}_{\kappa}^{n \times n} \) in some Pontryagin space \( \mathcal{P} \) is a pair \((A, \Gamma_z)\) consisting of a self-adjoint relation \( A \) in \( \mathcal{P} \) with a nonempty resolvent set \( \rho(A) \) and a corresponding \( \Gamma \)-field \( \Gamma_z \), that is, a family of mappings \( \Gamma_z : \mathbb{C}^n \to \mathcal{P} \), \( z \in \rho(A) \), which satisfy
\[
\Gamma_z = (I_{\mathcal{P}} + (z - \zeta)(A - z)^{-1})\Gamma_\zeta, \quad \zeta, z \in \rho(A),
\]
and
\[
\frac{N(\zeta) - N(z)^*}{\zeta - z^*} = \Gamma_z^*\Gamma_\zeta, \quad \zeta, z \in \rho(A), \quad z \neq \zeta^*.
\]
If a point \( z_0 \in \rho(A) \) is fixed this implies the following representation of \( N \):
\[
N(z) = N(z_0)^* + (z - z_0^*)\Gamma_{z_0}^*(I_{\mathcal{P}} + (z - z_0)(A - z)^{-1})\Gamma_{z_0}, \quad z \in \mathcal{D}(N).
\]
The function \( N \) is determined by the self-adjoint relation \( A \) in \( \mathcal{P} \) and the \( \Gamma \)-field \( \Gamma_z \) up to an additive constant hermitian \( n \times n \) matrix. The space \( \mathcal{P} \) is called the state space of the realization \((A, \Gamma_z)\). The realization \((A, \Gamma_z)\) can always be chosen minimal which means that
\[
\overline{\text{span}} \{ \Gamma_z c \mid z \in \rho(A), \, c \in \mathbb{C}^n \} = \mathcal{P}.
\]
In that case the negative index of the state space \( \mathcal{P} \) is equal to the number of negative squares of the kernel \( K_N(\zeta, z) \) and \( \mathcal{D}(N) = \rho(A) \); see [16, Theorem 1.1].

Two minimal realizations of \( N \) are unitarily equivalent. With a minimal realization \((A, \Gamma_z)\) often a symmetric restriction \( S \) of the relation \( A \) is associated and defined by
\[
S = \{ \{ f, g \} \in A \mid \Gamma_{z_0}^*(g - z_0^*f) = 0 \}.
\]
This definition is independent of \( z_0 \in \mathcal{D}(N) \), \( S \) is an operator, and \( \Gamma_z \) maps \( \mathbb{C}^n \) onto the defect subspace \( \text{ran}(S - z^*)^{-1} \) of \( S \) at \( z \). The triplet \((A, \Gamma_z, S)\) is called a model in \( \mathcal{P} \) for the realization of \( N \) or, for short, a model for the function \( N \) in \( \mathcal{P} \). The model will be called minimal if the realization is minimal.

If \( n = 1 \) the function
\[
\varphi(z) = \Gamma_z 1 = (I_{\mathcal{P}} + (z - z_0)(A - z)^{-1})\varphi(z_0),
\]
called a defect function for \( S \) and \( A \), spans the defect subspace of \( S \) at \( z \) and the representation of \( N \) takes the form
\[
N(z) = N(z_0)^* + (z - z_0^*)(\varphi(z), \varphi(z_0))_{\mathcal{P}}.
\]
Every \( N \in \mathcal{N}_{\kappa} \) admits a basic factorization of the form
\[
N(z) = r^\#(z)N_1(z)r(z), \quad (1.1)
\]