ANALYSIS OF HIGHER ORDER METHODS FOR
THE NUMERICAL SIMULATION OF CONFINED FLOWS

P. BONTOUX, B. FORESTIER and B. ROUX

Institut de Mécanique des Fluides de Marseille
13003 MARSEILLE - FRANCE

ABSTRACT

The compact differencing method developed by Hirsh is used to solve the two dimensional incompressible Navier-Stokes equations. The higher order of accuracy is investigated in the case of the test problem proposed by Pearson. Criteria for the stability of the ADI scheme are derived in terms of the physical parameters, when various types of boundary conditions are involved. Accelerating techniques are tested in order to optimize the convergence of the iterative process. Applications are made in the present paper for the driven cavity problem.

INTRODUCTION

In recent methods, the higher order of accuracy is obtained by using only three points finite difference discretization, when the derivatives of the function are considered as supplementary unknowns. The closure of the system is then ensured with additional relationships, called Hermitian formulae, which relate the values of the function and its two first derivatives at three mesh points. Three points differences have been already developed for a sixth order accuracy by Rubin and Khosla, 1977. The interest is actually centered on the construction of fourth order accurate schemes, as witnessed by the works published by Orszag and Israeli, 1974, Hirsh, 1975, Krause et al, 1976, and Adam, 1977. Peyret, 1978 recently studied the errors corresponding to these various methods for a simple linear differential equation, and showed that they are formally the smallest when implicit hermitian formulae are used for the second derivatives, which is in agreement with our own previous numerical experiments (Bontoux,
Forestier and Roux, 1977). The compact differencing method has then been used extensively as it is also quite readily applicable to the solution of the Navier Stokes equations with an ADI scheme.

**COMPACT DIFFERENCING METHOD**

The implicit hermitian relationship read as:

\[
(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1} = \frac{3}{h} (u_{i+1} - u_{i-1}) + O(h^4) \tag{1}
\]

\[
(u_{xx})_{i+1} + 10(u_{xx})_i + (u_{xx})_{i-1} = \frac{12}{h^2} (u_{i+1} - 2u_i + u_{i-1}) + O(h^4) \tag{2}
\]

The system includes the second order differential equation under the discretized form:

\[
m_{1i} u_i + m_{2i} (u_x)_i + m_{3i} (u_{xx})_i + m_{4i} = 0 \tag{3}
\]

When the boundary conditions are known for the function and its derivatives, the (3x3) block tridiagonal system may be solved by using the generalized Thomas algorithm. Usual boundary conditions are however of Dirichlet or Neumann types, and it is then necessary to consider one more additional relation between the values of the function and its derivatives on the boundary and in the field. Such relations which preserve the tridiagonal character of the equations, are only third order accurate. The Padé approximant has been retained:

\[
(u_{xx})_i - (u_{xx})_{i+1} + \frac{6}{h} ((u_x)_i + (u_x)_{i+1}) + \frac{12}{h^2} (u_i - u_{i+1}) = 0 + O(h^3) \tag{4}
\]

**GOVERNING EQUATIONS**

The present study is devoted to confined flows. The governing equations are considered with the vorticity, \( \zeta \), and the stream function, \( \psi \), as dependant variables. The unsteady Navier-Stokes equations are written under the convective form as follows:

\[
\zeta_t + u \zeta_x + v \zeta_y = a_1 (\zeta_{xx} + \zeta_{yy}) \tag{5a}
\]

\[
\psi_{xx} + \psi_{yy} = \zeta \tag{5b}
\]

where \( u = \psi_y \) and \( v = -\psi_x \). In the applications where \( \psi \) and its first derivative normal to the wall are given on each boundary, the conditions on \( \psi \) are overspecified. However, no physical condition exists then for the vorticity \( \zeta_w \) at the boundaries, and \( \zeta_w \) must be evaluated in terms of \( \psi \) from (5b) as: