SUMS OF SQUARES, MOMENT MATRICES AND OPTIMIZATION OVER POLYNOMIALS

MONIQUE LAURENT*

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Abstract. We consider the problem of minimizing a polynomial over a semialgebraic set defined by polynomial equations and inequalities, which is NP-hard in general. Hierarchies of semidefinite relaxations have been proposed in the literature, involving positive semidefinite moment matrices and the dual theory of sums of squares of polynomials. We present these hierarchies of approximations and their main properties: asymptotic/finite convergence, optimality certificate, and extraction of global optimum solutions. We review the mathematical tools underlying these properties, in particular, some sums of squares representation results for positive polynomials, some results about moment matrices (in particular, of Curto and Fialkow), and the algebraic eigenvalue method for solving zero-dimensional systems of polynomial equations. We try whenever possible to provide detailed proofs and background.

Key words. positive polynomial, sum of squares of polynomials, moment problem, polynomial optimization, semidefinite programming.


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*CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands. Supported by the Netherlands Organisation for Scientific Research grant NWO 639.032.203 and by ADONET, Marie Curie Research Training Network MRTN-CT-2003-504438.
1. Introduction. This survey focuses on the following polynomial optimization problem: Given polynomials $p, g_1, \ldots, g_m \in \mathbb{R}[x]$, find

$$p^{\min} := \inf_{x \in \mathbb{R}^n} p(x) \text{ subject to } g_1(x) \geq 0, \ldots, g_m(x) \geq 0, \quad (1.1)$$

the infimum of $p$ over the basic closed semialgebraic set

$$K := \{ x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}. \quad (1.2)$$

Here $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ denotes the ring of multivariate polynomials in the $n$-tuple of variables $x = (x_1, \ldots, x_n)$. This is a hard, in general non-convex, optimization problem. The objective of this paper is to survey relaxations methods for this problem, that are based on relaxing positivity over $K$ by sums of squares decompositions, and the dual theory of moments. The polynomial optimization problem arises in numerous applications. In the rest of the Introduction, we present several instances of this problem, discuss the scope of the paper, and give some preliminaries about polynomials and semidefinite programming.