Chapter 1
QR Decomposition: An Annotated Bibliography

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Abstract This chapter is divided into two parts. The first one goes back in time and tries to retrace the steps of great mathematicians who laid the foundations of numerical linear algebra. We describe some early methods to compute the eigenvalues and eigenvectors associated to a matrix $A$. The QR decomposition (orthogonalization as in Gram–Schmidt) is encountered in many of these methods as a fundamental tool for the factorization of $A$. The first part describes the QR algorithm, which uses the QR decomposition iteratively for solving the eigenproblem $Ax = \lambda x$. The second part of the chapter analyzes the application of the QR decomposition to adaptive filtering.

1.1 Preamble

To tell the story of the QR decomposition, we must go back in history, to an era when electronic calculating machines were first built and the associated algorithmic programming languages were first proposed [1, 2]. Two problems that were common to different applications and of particular importance to the newly created field of applied mathematics were: The solution of large systems of linear simultaneous equations

$$Ax = b,$$

(1.1)
often arising as consequence of least-squares minimization [3], and the solution of the eigenvalue–eigenvector problem [4, 5]

\[ \mathbf{A}\mathbf{x} = \lambda \mathbf{x}. \]  

These problems may also be seen as particular cases of the problem of Fredholm [6] in the matrix format (see, e.g., [7]),

\[ \mathbf{x} - \lambda \mathbf{A}\mathbf{x} = \mathbf{b}, \]  

which simplifies to the eigenproblem for \( \mathbf{b} = \mathbf{0} \) and to the solution of the linear system of equations for \( \lambda \to \infty \). The methods to be outlined here are systematic procedures to solve the problem, offering a more economical and sometimes a more stable alternative to matrix inversion or to the Liouville-Neumann expansion

\[ \mathbf{x} = (\mathbf{I} - \lambda \mathbf{A})^{-1} \mathbf{b} = (\mathbf{I} + \lambda \mathbf{A} + \lambda^2 \mathbf{A}^2 + \cdots)\mathbf{b}. \]  

(1.4)

For an account of the early developments in numerical linear algebra related to the different methods for matrix inversion and eigenproblem solution see, e.g., [8–11].

In the following sections, the QR decomposition and the QR algorithm, which have become a standard subject in all linear algebra books (see, e.g., [3, 12–16]), will be placed in the context of the solutions proposed for the Equation (1.2). We will describe the solutions trying to give credit to key researchers that contributed to the development of the methods, as well as their refinement and dissemination.

### 1.2 Eigenvalues and Eigenvectors

The eigenproblem, as the eigenvalue–eigenvector problem is usually referred to, can be traced back more than 150 years ago to the pioneering work of Jacobi [17] (see also [18–21]). He has not described the problem in matrix notation, which was invented shortly after [22–24]. Jacobi indeed proposed an ingenious solution to the Equation (1.2) for the particular case of symmetric matrices.

A non-zero vector \( \mathbf{x} \in \mathbb{C}^N \) is an eigenvector of \( \mathbf{A} \in \mathbb{C}^{N \times N} \) and \( \lambda \in \mathbb{C} \) is the associated eigenvalue if the Equation (1.2) is true. Since \( (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} \), the eigenvalues are the roots of the characteristic polynomial, given by

\[ \wp_{\mathbf{A}}(z) = \det(z\mathbf{I} - \mathbf{A}), \]  

(1.5)

where \( \mathbf{I} \) is the identity matrix [25]. A scalar \( \lambda \) is an eigenvalue of \( \mathbf{A} \) if and only if \( \wp_{\mathbf{A}}(\lambda) = 0 \) [19, 26]. But the roots of an \( N \)th-degree polynomial have no closed-form formula for \( N > 4 \) [27–29], and a direct approach yields an ill-conditioned problem. Therefore the solution of the eigenproblem must resort to iterative methods that reveal the eigenvalues of \( \mathbf{A} \) as they factor or transform the matrix.