Chapter 7

The Riemann Integral

7.1 Discussion: How Should Integration be Defined?

The Fundamental Theorem of Calculus is a statement about the inverse relationship between differentiation and integration. It comes in two parts, depending on whether we are differentiating an integral or integrating a derivative. Under suitable hypotheses on the functions $f$ and $F$, the Fundamental Theorem of Calculus states that

(i) $\int_a^b F'(x) \, dx = F(b) - F(a)$ and

(ii) if $G(x) = \int_a^x f(t) \, dt$, then $G'(x) = f(x)$.

Before we can undertake any type of rigorous investigation of these statements, we need to settle on a definition for $\int_a^b f$. Historically, the concept of integration was defined as the inverse process of differentiation. In other words, the integral of a function $f$ was understood to be a function $F$ that satisfied $F' = f$. Newton, Leibniz, Fermat, and the other founders of calculus then went on to explore the relationship between antiderivatives and the problem of computing areas. This approach is ultimately unsatisfying from the point of view of analysis because it results in a very limited number of functions that can be integrated. Recall that every derivative satisfies the intermediate value property (Darboux's Theorem, Theorem 5.2.7). This means that any function with a jump discontinuity cannot be a derivative. If we want to define integration via antidifferentiation, then we must accept the consequence that a function as simple as

$$h(x) = \begin{cases} 
1 & \text{for } 0 \leq x < 1 \\
2 & \text{for } 1 \leq x \leq 2
\end{cases}$$

is not integrable on the interval $[0, 2]$. 

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A very interesting shift in emphasis occurred around 1850 in the work of Cauchy, and soon after in the work of Bernhard Riemann. The idea was to completely divorce integration from the derivative and instead use the notion of “area under the curve” as a starting point for building a rigorous definition of the integral. The reasons for this were complicated. As we have mentioned earlier (Section 1.2), the concept of function was undergoing a transformation. The traditional understanding of a function as a holistic formula such as \( f(x) = x^2 \) was being replaced with a more liberal interpretation, which included such bizarre constructions as Dirichlet’s function discussed in Section 4.1. Serving as a catalyst to this evolution was the budding theory of Fourier series (discussed in Section 8.3), which required, among other things, the need to be able to integrate these more unruly objects.

The Riemann integral, as it is called today, is the one usually discussed in introductory calculus. Starting with a function \( f \) on \([a, b]\), we partition the domain into small subintervals. On each subinterval \([x_{k-1}, x_k]\), we pick some point \( c_k \in [x_{k-1}, x_k] \) and use the \( y \)-value \( f(c_k) \) as an approximation for \( f \) on \([x_{k-1}, x_k]\). Graphically speaking, the result is a row of thin rectangles constructed to approximate the area between \( f \) and the \( x \)-axis. The area of each rectangle is \( f(c_k)(x_k - x_{k-1}) \), and so the total area of all of the rectangles is given by the Riemann sum (Fig. 7.1)

\[
\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).
\]

Note that “area” here comes with the understanding that areas below the \( x \)-axis are assigned a negative value.

What should be evident from the graph is that the accuracy of the Riemann-sum approximation seems to improve as the rectangles get thinner. In some