5

The Representations of SU(3)

5.1 Introduction

There is a theory of the representations of semisimple groups and Lie algebras (discussed in Chapters 6, 7, and 8) that includes as a special case the representation theory of SU(3). However, I feel that it is worthwhile to examine the case of SU(3) separately, before going on to the general theory. I feel this way partly because SU(3) is an important group in physics, but chiefly because the general semisimple theory is difficult to digest. Considering a nontrivial example makes what is going on much clearer. In fact, all of the elements of the general theory are present already in the case of SU(3), so we do not lose too much by considering at first just this case.

The main result of this chapter is Theorem 5.9, which states that an irreducible finite-dimensional representation of SU(3) can be classified in terms of its "highest weight." This is analogous to labeling the irreducible representations $V_m$ of SU(2) or sl(2; $\mathbb{C}$) by the highest eigenvalue of $\pi_m(H)$. (The highest eigenvalue of $\pi_m(H)$ in $V_m$ is precisely $m$.) In the next two chapters, we will look at the analogous results for general semisimple Lie algebras.

The group SU(3) is simply connected (Appendix E), and so the finite-dimensional representations of SU(3) are in one-to-one correspondence with the finite-dimensional representations of the Lie algebra su(3). Meanwhile, the complex representations of su(3) are in one-to-one correspondence with the complex-linear representations of the complexified Lie algebra su(3)$_C$ = sl(3; $\mathbb{C}$) (Proposition 4.6). Moreover, a representation of SU(3) is irreducible if and only if the associated representation of su(3) is irreducible, and this holds if and only if the associated complex-linear representation of sl(3; $\mathbb{C}$) is irreducible. (This follows from Proposition 4.5, Proposition 4.6, and the connectedness of SU(3).) Thus, we have the following result.

Proposition 5.1. There is a one-to-one correspondence between the finite-dimensional complex representations $\Pi$ of SU(3) and the finite-dimensional
complex-linear representations $\pi$ of $\mathfrak{sl}(3; \mathbb{C})$. This correspondence is determined by the property that

$$\pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{su}(3) \subset \mathfrak{sl}(3; \mathbb{C})$.

The representation $\Pi$ is irreducible if and only if the representation $\pi$ is irreducible.

Since $SU(3)$ is compact, Proposition 4.36 tells us that all of the finite-dimensional representations of $SU(3)$ are direct sums of irreducible representations. The above proposition then implies that the same holds for $\mathfrak{sl}(3; \mathbb{C})$; that is, $\mathfrak{sl}(3; \mathbb{C})$ has the complete reducibility property. Complete reducibility will be an essential ingredient even in the classification of irreducible representations. (See the proof of Proposition 5.16.)

Moreover, we can apply the same reasoning to the simply-connected group $SU(2)$, its Lie algebra $\mathfrak{su}(2)$, and its complexified Lie algebra $\mathfrak{sl}(2; \mathbb{C})$. Thus, we have established the following.

**Proposition 5.2.** Every finite-dimensional representation of $\mathfrak{sl}(2; \mathbb{C})$ or $\mathfrak{sl}(3; \mathbb{C})$ decomposes as a direct sum of irreducible invariant subspaces.

We will use the following basis for $\mathfrak{sl}(3; \mathbb{C})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that the span of $\{H_1, X_1, Y_1\}$ is a subalgebra of $\mathfrak{sl}(3; \mathbb{C})$ which is isomorphic to $\mathfrak{sl}(2; \mathbb{C})$ (as can be seen by ignoring the third row and the third column in each matrix). Similarly, the span of $\{H_2, X_2, Y_2\}$ is a subalgebra isomorphic to $\mathfrak{sl}(2; \mathbb{C})$. Thus, we have the following commutation relations:

$$[H_1, X_1] = 2X_1, \quad [H_2, X_2] = 2X_2,$$
$$[H_1, Y_1] = -2Y_1, \quad [H_2, Y_2] = -2Y_2,$$
$$[X_1, Y_1] = H_1, \quad [X_2, Y_2] = H_2.$$

We now list all of the commutation relations among the basis elements which involve at least one of $H_1$ and $H_2$. (This includes some repetitions of the above commutation relations.)