In this chapter we start to develop methods for the study of sharp asymptotics under the sequence Gaussian model. These methods are based on the study of likelihood ratio statistics in Bayesian problems and on the study of the minimax properties of tests based on these statistics.

We would like to show that for a wide class of hypothesis testing problems there exists a duality between the choice of an asymptotically best product prior and the selection of asymptotically best tests. Namely, both problems lead to the extreme problem: to find "nearest to zero point" over a specific convex set in a Hilbert space.

First we consider the simplest case: Bayesian priors are Dirac masses concentrated on points in the alternative. Here the Bayesian likelihood ratio statistics lead to tests based on linear statistics and, if an alternative set is convex, then the selection of the best tests, based on linear statistics, is dual (nonasymptotically) to the choice of the best prior. We study in detail the extreme problem for the positive alternative determined by two-sided constraints for the power norms and for the norms (2.70) defined by the sequences of exponential type with $p \leq 1$, $q \geq p$.

The reason of this study is that the analogous extreme problems we obtain in general cases. In particular, this duality is extended to symmetric two-point product priors and tests of $\chi^2$-type. These provide a translation of results from positive alternatives to general alternatives determined by two-sided constraints for the power and Besov norms with $p \leq 2$, $q \geq p$.

The study of the relationships between orthogonal priors and supreme-tests provide a description of the regions of the asymptotics of degenerate type.
4.1 Tests Based on Linear Statistics and Convex Alternatives

In this and the next section we will study sharp asymptotics for special classes of infinite-dimensional alternatives under convex assumptions (see Section 2.4.3 above). In this case the minimax hypothesis testing problem is reduced to a special extreme problem. Our aim is to describe the key ideas of duality and analytical methods to study the extreme problem which will be generalized below for a wide class of minimax hypothesis testing problems under an asymptotic approach.

Under the sequence Gaussian model (2.1) let us test $H_0: v = 0$ against $H_1: v \in V \subset l^2$. One can write simple lower bounds in the problem. Denote

$$u = \inf_{v \in V} |v|. \quad (4.1)$$

For any $v \in V$ we have (see Example 2.1)

$$\beta(\alpha, V) \geq \beta(\alpha, \{v\}) = \Phi(T\alpha - |v|), \quad \gamma(V) \geq \gamma(\{v\}) = 2\Phi(-|v|/2).$$

By taking supremum we get

$$\beta(\alpha, V) \geq \Phi(T\alpha - u), \quad \gamma(V) \geq 2\Phi(-u/2). \quad (4.2)$$

The problem is: Are the lower bounds (4.2) good enough? Sometimes we have a positive answer.

Consider tests based on linear statistics $\psi_{r,T} = 1_{\{<x, r> > T\}}$ based on the linear statistics

$$t_r = (x, r) = \sum_i x_i r_i, \quad r \in l^2; \quad |r|^2 = |r|^2 = (r, r) = 1,$$

and $T$ is a nonrandom threshold. It was shown in Section 3.1.1 that

$$\alpha(\psi_{r,T}) = \Phi(-T), \quad \beta(\psi_{r,T}, V) = \sup_{v \in V} \Phi(T - (r, v)) = \Phi(T - h(r, V)),$$

where

$$h(r, V) = \inf_{v \in V} (r, v)$$

is the minimum of the projection of $v \in V$ to the direction $r$. Thus to find the best linear test we need to choose the direction $r$ which maximizes $h(r, V)$. This corresponds to the maximin problem

$$h = h(V) = \sup_{r \in l^2: |r| = 1} h(r, V) = \sup_{r \in l^2: |r| = 1} \inf_{v \in V} (r, v). \quad (4.3)$$

Suppose $h > 0$. Then one can easily see that it is possible to change the constraint $|r| = 1$ by the convex constraint $|r| \leq 1$. Assume the set $V$ is convex. Then, using the equality

$$\sup_{r \in l^2: |r| \leq 1} (r, v) = |v|$$