1. Introduction. This paper is about the definition and effectiveness and fast implementation of a particular family of smoothing filters. These “Savitzky-Golay filters” are popular in spectroscopy. But to filter experts in other areas they are virtually unknown! Since the filters are constructed in a very natural way, they allow analysis and explanation. They can give excellent results provided the filter length is correctly chosen, and they deserve to be understood.

Their construction is based on a least squares fit to each window of data by a polynomial of fixed degree \( n \). The (smoothed) output value is taken at the center of the window. Already this may raise doubts. Starting from an ideal lowpass filter (with one-zero frequency response), its least squares approximation is not a successful favorite. We would be truncating the slowly convergent Fourier series of a step function (with Gibbs phenomenon at the jump). Minimizing the maximum error produces equiripple filters that generally perform better. On the other hand equiripple filters don’t preserve moments of the input signal, which spectroscopists want. Savitzky-Golay fits the data by low-degree polynomials in the time domain, not high-degree polynomials in the frequency domain. The construction is robust for long filters, so the window can and should match the intrinsic scale of the input signal.

We will give explicit formulas for the filter coefficients (of course the formulas are well established for polynomials of low degree \( n \)). For all degrees, we show how the Savitzky-Golay filters come directly from Chebyshev’s construction in 1854 of “discrete orthogonal polynomials”. The most direct approach is to orthogonalize the \( n+1 \) vectors \((1,1,\ldots,1),(0,1,\ldots,N-1),\ldots,(0^n,1^n,\ldots,(N-1)^n)\), which are the columns of a rectangular Vandermonde matrix. Least squares is simplified, as always, by orthogonality. The polynomials satisfy a three-term recurrence, and a Christoffel-Darboux sum formula. All the classical properties extend to these polynomial vectors \( t_n \), and lead to concise formulas for the filters.

The continuous analogue of Chebyshev’s construction produces the Legendre polynomials. (It is Gram-Schmidt on the functions \(1, x, x^2, \ldots, x^n\). The inner product is an integral instead of a sum.) Naturally the limit of the discrete polynomial vectors, with suitable scaling, is Legendre’s family of continuous polynomials \( P_n(x) \). If we sample these polynomials, we are extremely close to Savitzky-Golay. In fact, for reasons of simplicity...
and speed, we recommend the Legendre-based filters (you will see that they represent the leading terms in the Chebyshev-Savitzky-Golay formulas). Legendre has the extra advantage, in the case of irregularly spaced or missing data, that the polynomials stay the same and it is only the sampling points that change.

Our forthcoming paper [9] will provide software in pseudo-code to execute both fast transforms (Savitzky-Golay and Legendre-based). The normal implementation of a length $N$ filter involves $N$ multiplications in each window. But a polynomial of degree $n$ is determined by $n + 1$ coefficients. Bromba and Ziegler in [1] and Scott and Scott in [11] have shown how each output value requires only $O(n + 1)$ steps, using the previous outputs recursively. For degree $n = 0$, the filter is a simple average (mean filter) over the window. Shifting the window adds one new sample and drops one old sample. The average over that new window updates the previous average by using those two samples:

\begin{equation}
\text{new average} = \text{old average} + \left( x_{\text{newest}} - x_{\text{oldest}} \right) / N.
\end{equation}

For a higher $n$, the recursive correction involves polynomial degrees lower than $n$ (and stability is a significant problem). It is fast multiplication by Toeplitz matrices with “polynomial rows”.

The important question, and the hardest to answer, is the effectiveness of these filters. We will report on experiments that largely justify their use (for a correctly chosen filter length!) in a significant class of applications. Our simplest model is a single Gaussian corrupted by white noise. In this case we can analyze the standard rule that the filter window should match the width of the Gaussian at half maximum. A correction term can reflect the degree $n$. By comparing Savitzky-Golay and Legendre with equiripple, the reader can see how to match the choice of filter with the application.

In summary, the four parts of our final paper [9] will attempt to provide:

1. Explicit formulas for Savitzky-Golay from Chebyshev’s discrete orthogonal polynomials $q_n(x)$. The filter’s impulse response (a multiple of $q_{n+1}(k)/k$) is the least squares approximation of a discrete delta function.
2. New Legendre-based polynomial filters from a continuous analogue of the same construction. Applying the Christoffel-Darboux sum formula leads us to propose symmetric filters of even degree $n$ formed by sampling a multiple of Legendre’s $P_{n+1}(x)/x$.
3. Numerical experiments and analysis for “Gaussian signals plus noise” to determine the characteristics of these filters. We could study also the asymptotics, for large (and moderate) $N$ and $n$, in the time and frequency domains.
4. Fast implementation of both filters by a stabilized recursion with $O(n + 1)$ steps per output sample.