SPECTRAL THEORY FOR NEUTRAL DELAY EQUATIONS WITH APPLICATIONS TO CONTROL AND STABILIZATION

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Abstract. For dynamical systems governed by feedback laws, time delays arise naturally in the feedback loop to represent effects due to communication, transmission, transportation or inertia effects. The introduction of time delays in a system of differential equations results in an infinite dimensional state space. The solution operator associated with a differential delay equation is a nonself-adjoint operator defined on a Banach space. This implies that general abstract theorems cannot directly be applied. In this paper we discuss the spectral properties of neutral differential delay equations, series expansions of solutions, completeness of eigenvectors and generalized eigenvectors, solutions of neutral delay equations that decay faster than any exponential, and applications to control and stabilization.

Key words. completeness, F-completeness, neutral delay equation, robustness, sensitivity, small solutions, small delays, stabilizability.

AMS(MOS) subject classifications. Primary 34K.

1. Introduction. The aim of this paper is to present the spectral theory for difference equations and neutral delay equations. We start with an introduction to the theory of delay equations. The basic topics include the very definition, the state space approach and the solution operator. After this introduction, we turn to an analysis of the spectral properties of difference equations and neutral differential delay equations. Not only are differential delay equations a rich class of nonself-adjoint problems with very interesting spectral issues which serve as motivation for a general theory (cf. [18, 49, 51]). They are also important in applications (cf. [15, 19] and the references given there). As motivation for the theory that we develop in this paper, we briefly discuss some examples.

In the implementation of any feedback control system, e.g., the control of partial differential equations through the application of forces on the boundary, it is very likely that time delays will occur [2]. Therefore, it is of importance to understand the sensitivity of the control system with respect to the introduction of small delays in the feedback loop. For some systems, small delays lead to destabilization while other systems are robust with respect to small time delays. In [20] a first attempt was made for a unifying theory which explains the underlying mechanisms in terms of spectral properties of the solution operator. In order to illustrate the theory that we will develop in Section 6, we first discuss an example.

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Consider the difference equation

\begin{equation}
(1.1) \quad x(t) - a_1 x(t - r_1) - a_2 x(t - r_2) = 0,
\end{equation}

where \( r_1 \) and \( r_2 \) are rationally independent. According to Corollary 4.3 of this paper, this equation is not uniformly exponentially stable if and only if

\[ |a_1| + |a_2| \geq 1. \]

Given \( |a_1| + |a_2| \geq 1 \), we can apply a feedback control to stabilize (1.1)

\begin{equation}
(1.2) \quad \dot{x}(t) - (a_1 + f_1)x(t - r_1) - (a_2 + f_2)x(t - r_2) = 0.
\end{equation}

This closed-loop system is exponentially stable if and only if

\[ |a_1 + f_1| + |a_2 + f_2| < 1. \]

Suppose now that the feedback control cannot be applied instantaneously and that there is a small time delay in the feedback loop. This leads to the following equation

\begin{equation}
(1.3) \quad \dot{x}(t) - \sum_{j=1}^{2} a_j x(t - r_j) - \sum_{j=1}^{2} f_j x(t - r_j - \epsilon_j) = 0.
\end{equation}

We claim that, although (1.3) is exponentially stable for \( \epsilon_1 = \epsilon_2 = 0 \), there is a sequence \( (\epsilon_1, \epsilon_2) \) tending to zero so that (1.3) is exponentially unstable. To prove the claim, we choose \( \epsilon_1 \) and \( \epsilon_2 \) such that \( (\epsilon_1, \epsilon_2) \rightarrow (0,0) \) and \( r_1, r_2, r_1 + \epsilon_1, \) and \( r_2 + \epsilon_2 \) are rationally independent. From Corollary 4.3, it follows that (1.3) is always exponentially unstable, since

\[ |a_1| + |f_1| + |a_2| + |f_2| > |a_1| + |a_2| \geq 1. \]

This example shows that feedback control might be sensitive to the delay and in Section 6 we will discuss these issues.

The second example concerns the identifiability of unknown parameters that appear in special classes of nonself-adjoint evolutionary systems. Parameter identifiability is concerned with the question whether the parameters of a specific model can be identified from knowledge about certain solutions of the model, assuming perfect data. Completeness of the eigenvectors and generalized eigenvectors of the solution operator plays a crucial role in these results ([52, 53]). In order to illustrate the main theorem, we consider the identification problem for the following scalar neutral delay equation

\begin{equation}
(1.4) \quad \frac{d}{dt} \left[ y(t) - \sum_{j=1}^{k} d_j y(t - r_j) - \int_{-h}^{0} d_1(s)y(t + s) \, ds \right] = \int_{-h}^{0} d_2(\theta)y(t + \theta),
\end{equation}