NUMERICS VERSUS CONTROL

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Abstract. Numerical matrix eigenvalue methods such as the inverse power iteration or the QR-algorithm can be reformulated as inverse power iterations on homogeneous spaces. In this paper we survey some recent results on controllability properties of the shifted inverse power iteration on flag manifolds. It is shown that the reachable sets are orbits for a semigroup action on the flag manifold. Except for the special case of projective spaces, the algorithm is never controllable. This implies in particular the non-controllability of the shifted QR-algorithm on isospectral matrices. Controllability results for the inverse power iteration on projective space for real or complex shifts are presented, following [20, 22], and a connection with output feedback pole assignability is mentioned. Controllability of the algorithm on Hessenberg flags is shown. This implies controllability of the shifted QR-algorithm on Hessenberg matrices.

Key words. Inverse iteration, Hessenberg matrices, controllability, reachable sets, flag manifolds, QR-algorithm.

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1. Introduction. A lot of the current interaction between numerical analysis and control theory is focussed on the development of numerical algorithms for control systems. The converse direction, that of applying control theory to the analysis and design of numerical algorithms, has been a less travelled road. A key observation is that most iterative numerical algorithms, such as e.g. matrix eigenvalue methods, can be interpreted as nonlinear discrete-time dynamical systems evolving on manifolds. Interesting examples of such an approach can be found in the work by Ammar and Martin [3], Batterson and Smillie [5, 6], Deift, Nanda et al. [12], Flaschka [16] and Shub and Vasquez [25] in the 1980s, where the dynamics of the QR-algorithm or inverse power iterations are analyzed using tools from dynamical systems theory. To proceed in a different direction, gradient flow techniques or Hamiltonian systems can help to find new numerical algorithms, thus again stressing the importance of dynamical systems methods for computation; see e.g. Brockett [9–11], Bloch et al. [7, 8], Edelman, Arias and Smith [15], and Helmke and Moore [21].

Control theory has an important role to play in this circle of ideas. We mention only two different research directions here, although there are probably several more of interest. One central area is that of step-size control for ODE-solvers. The step-size being an important control parameter to be tuned to improve the discretization error. Basic techniques from control can be successfully applied to optimize the discretization error, see e.g. Gustafsson [19] for work in this direction. Another area of interest concerns the control of matrix eigenvalue algorithms. Standard eigenvalue
methods, such as the QR-, or more generally FG-algorithms, or inverse power iterations are usually only linearly convergent and thus are too slowly convergent to be of practical use. Therefore eigenvalue shift strategies have been introduced to speed up convergence. Such spectral shift parameters act as control variables in the system and the resulting algorithms can thus be analyzed from the viewpoint of nonlinear control theory. The analysis and design of these shifted eigenvalue methods is however a difficult and only partially understood task. For example, in the book *Open Problems in Mathematical Systems and Control Theory*, Van Dooren and Sepulchre [26] mention it as an open problem for the general class of FG-algorithms. Specific convergence results on the inverse power iteration with Rayleigh quotient shifts can be found e.g. in [5, 6]. More recently, a multishift version of the QR-algorithm has been proposed by Absil, Mahony, Sepulchre and Van Dooren [1], see also the recent work by Hüper [23, 24].

In earlier work [20, 22], we have analyzed the controllability properties of the inverse power iteration on projective space and Grassmann manifolds. The purpose of the present paper is to survey these and other recent results on the controllability of inverse iteration schemes on general flag manifolds and the manifold of Hessenberg flags.

We begin by describing the QR-algorithm on the partial flag manifold. It is shown that the algorithm is never controllable, except for special cases. Controllability results for the inverse power iteration on projective space with real or complex shifts are presented, following the works [20, 22]. The case of real shifts is quite complicated and only partial results are known so far. An interesting connection with real output feedback pole assignment is mentioned. Fundamental limitations of any feedback strategy for eigenvalue shifts are imposed by the closures of the reachable sets. Thus a combinatorial description of such closures in flag manifolds is given, making contact with the work by Gelfand et al. [17] on Grassmann simplices. Finally, it is shown that the shifted QR-algorithm on Hessenberg matrices is controllable by showing the controllability of the inverse iteration scheme on Hessenberg flags. Controllability of the shifted QR-algorithm on real symmetric isospectral tridiagonal matrices has also been shown independently by G. Gladwell (unpublished work) and presented in a lecture at the 4th SIAM Conference on Linear Algebra in Signals, Systems and Control, Boston, 2001. Gladwell's approach is quite different to ours and is based on the preservation of total positivity of tridiagonal matrices under the QR-algorithm; see [18].

2. Inverse iteration on flag manifolds. The analysis of the shifted QR-algorithm is considerably simplified by viewing it as an algorithm evolving on subspaces rather than on vector tuples. This leads to the familiar interpretation of the QR-algorithm on the flag manifold; see e.g. [3].