ON THE APPROXIMATION OF COMPLICATED
DYNAMICAL BEHAVIOR*

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Abstract. We present efficient techniques for the numerical approximation of complicated
dynamical behavior. In particular, we develop numerical methods which allow us to approximate
Sinai–Ruelle–Bowen (SRB)-measures as well as (almost) cyclic behavior of a dynamical system.
The methods are based on an appropriate discretization of the Frobenius–Perron operator, and two
essentially different mathematical concepts are used: our idea is to combine classical convergence
results for finite dimensional approximations of compact operators with results from ergodic theory
concerning the approximation of SRB-measures by invariant measures of stochastically perturbed
systems. The efficiency of the methods is illustrated by several numerical examples.

Key words. computation of invariant measures, approximation of the Frobenius–Perron oper­
ator, computation of SRB-measures, almost invariant set, cyclic behavior

AMS subject classifications. 58F11, 65L60, 58F12

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1. Introduction. The approximation of the behavior of a dynamical system
is typically done by direct simulation. This method is particularly useful in the
situation where a specific trajectory has to be approximated for a finite period of
time. However, if one is interested in the long term behavior and if the underlying
system exhibits complicated dynamics, then the information derived from a single
trajectory is not always satisfying. Rather in this case it seems more appropriate to
determine a statistical description of the dynamical behavior, and this information is
encoded in an underlying (natural) invariant measure.

In this paper we describe a numerical method for the approximation of such in­
variant measures based on a discretization of the Frobenius–Perron operator. Using
the fact that invariant measures are fixed points of this operator, we first approximate
it by a Galerkin projection and then compute eigenvectors of the discretized operator
corresponding to the eigenvalue 1. This allows us to identify regions in state space
where trajectories are likely to be observed or, on the other hand, hardly observed.
In addition to this information we show how to use other parts of the spectrum of the
Frobenius–Perron operator to determine the dynamical behavior of the system. First
we describe how to decompose an invariant set into components which are cyclically
permuted by the dynamics. Second we develop techniques for the approximation of al­
most invariant sets, that is, regions in state space which are visited for a “long” period
of time before the dynamical process leads to different areas. More generally the same
techniques allow us to detect almost cyclic behavior, that is, to identify components of
invariant sets which are “frequently” cyclically permuted by the dynamical process.
Moreover, we can quantify the probability by which the cycle occurs depending on
the absolute value of a corresponding eigenvalue of the Frobenius–Perron operator.
Roughly speaking, we construct an approximation of the essential dynamical behavior,
that is, the dynamics modulo complex (unpredictable) behavior which is due to the presence of chaos.

From a practical point of view the most important invariant measures are the so-called SRB-measures. The reason is that for these measures the spatial and temporal averages of observables are identical for a set of initial conditions which has positive Lebesgue measure. The introduction of the underlying concept goes back to Y. Sinai (see [27]), and the existence of these measures has been shown for Axiom A systems by D. Ruelle and R. Bowen [26, 3]. In this article we suggest a numerical method for the approximation of SRB-measures, and in this context their stochastic stability is particularly important: first we use this fact as an analytical tool in our main convergence result in section 4, and second it is of practical importance if we view the numerical approximation as a small random perturbation. Indeed, stochastic stability of SRB-measures is guaranteed for Axiom A systems [17, 18].

More precisely there are two essential mathematical ingredients which allow us to develop a numerical method for the approximation of SRB-measures. We use a result of Yu. Kifer on the convergence of invariant measures in stochastic perturbations of the underlying dynamical system to the SRB-measure (see [17]) and combine this with results on the convergence of eigenspaces of discretized compact operators (see, e.g., [25]). The same technique is used for the approximation of the subsets in state space which are (almost) cyclically permuted by the dynamical process. With respect to the approximation of SRB-measures a similar result has previously been obtained by F. Hunt (see [15]). However, our methods are quite different from the ones used in that work. In particular, the results stated here cover the important situation where the random perturbations have a probability distribution with local support. In fact, this is the relevant case having in mind that the round off error in the numerical approximation can be interpreted as such a local perturbation. Another approach for the computation of SRB-measures—avoiding the approximation of the Frobenius–Perron operator—has recently been suggested by Yu. Kifer [19].

As mentioned above, in addition to the approximation of SRB-measures the main development in this paper is a numerical method which allows us to identify (almost) cyclic behavior. To accomplish this we use a Galerkin method to discretize the Frobenius–Perron operator in such a way that the discretization has the same cyclic properties as the operator itself. More precisely, if the underlying dynamical system has a cycle of order \(r\), then the \(r\)th roots of unity are eigenvalues of the Frobenius–Perron operator, and we will show that the corresponding eigenmeasures \(\nu_0, \ldots, \nu_{r-1}\) yield the desired information on the cyclic components: these components can be identified as supports of probability measures obtained by specific linear combinations of \(\nu_0, \ldots, \nu_{r-1}\). Our Galerkin approximation respects the cyclic behavior in the sense that the \(r\)th roots of unity are also eigenvalues of the discretized operator and that the corresponding eigenvectors converge to the eigenmeasures \(\nu_0, \ldots, \nu_{r-1}\) with increasing dimension of the approximating space. We will illustrate how to use these results to determine the subsets in state space which are almost cyclically permuted.

Finally, let us remark that the results on the approximation of the essential dynamical behavior obtained in this article can also be used to compute other statistical quantities such as the entropy, dimensions (depending on the particular invariant measure), or Lyapunov exponents. In fact, the efficient numerical use of invariant measures for the computation of Lyapunov exponents is currently under investigation.

An outline of the paper is as follows. In section 2 we begin with a brief review of the results on Markov processes which will be needed later on. The Frobenius–Perron