Dynamics of a System of Particles

8.1. Introduction

The principles of mechanics for a particle are extended here to a system of \( n \) discrete material points. We begin with Newton’s second law for a system of particles and formulate the momentum, impulse–momentum, moment of momentum, work–energy, conservation, and general energy principles for a system of particles. Several of the concepts introduced here are especially useful in the study (in Chapter 11) of Lagrange’s general equations for arbitrary dynamical systems, and the development of the moment of momentum principle for a system of particles provides a foundation for the independent presentation (in Chapter 10) of parallel results for the moment of momentum of a rigid body.

8.2. Equation of Motion for the Center of Mass

The total force \( F_k = F(P_k, t) \) that acts on the \( k \)th particle of a system \( \beta = \{ P_i \} \) of \( n \) particles consists of a total external force \( f_k = f(P_k, t) \) exerted by bodies outside of \( \beta \) and a total internal force \( b_k = b(P_k, t) \) due to the mutual interaction between \( P_k \) and all other particles in \( \beta \). Let \( b_{kj} \) denote the mutual internal force exerted on the particle \( P_k \) by the particle \( P_j \). Then the total internal force on \( P_k \) is

\[
b_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} b_{kj}.
\]  

(8.1)
Thus, the total force $F(\beta, t) = \sum_{k=1}^{n} F_k = \sum_{k=1}^{n} (f_k + b_k)$ acting on the system is

$$F(\beta, t) = \sum_{k=1}^{n} f_k + \sum_{k=1}^{n} \sum_{j \neq k}^{n} b_{kj}. \quad (8.2)$$

In accordance with the third law, the internal forces occur in equal, oppositely directed pairs so that

$$b_{jk} = -b_{kj}; \quad (8.3)$$

and hence the total internal force, the last sum in (8.2), vanishes. Therefore, the total force that acts on a system of particles is equal to the total external force:

$$F(\beta, t) = \sum_{k=1}^{n} f_k. \quad (8.4)$$

We recall from (5.7) that the total momentum of a system of particles is equal to the momentum of its center of mass, and use of this result in (5.40) leads to the familiar classical form (5.41) of Newton’s second law of motion for a system of particles in which only external forces (8.4) arise.

**Newton’s principle of motion for a system of particles:** The total external force on a system of particles is equal to the time rate of change of the momentum of the center of mass relative to an inertial frame $\Phi$, and is thus equal to the product of the total mass of the system and the acceleration of the center of mass in $\Phi$:

$$F(\beta, t) = \dot{P}^* (\beta, t) = m(\beta) a^*(\beta, t). \quad (8.5)$$

This equation aids in determination of the motion of the center of mass of the system and the external forces that act on it. In applications, however, the auxiliary center of mass relations (5.5) through (5.8), as well as the separate equations of motion of the particles, often are needed in problem solutions. Of course, the motion of an individual particle is governed by (5.39), which depends on the action of all forces that act on the particle, including internal forces that do not appear in (8.5). Without these auxiliary equations, (8.5) alone may not be very helpful. Plainly, all of the principles of mechanics for a single particle apply directly to the unique center of mass particle of a system of particles subjected to only the total external force (8.4). The familiar principle of conservation of momentum (7.69), for example, may be read immediately from (8.5), as follows below. Afterwards, an important application of the principle illustrates the need for the aforementioned auxiliary equations for the center of mass and Newton’s law for a particle.

**The principle of conservation of momentum of a system of particles:** The total external force component in a fixed direction $e$ vanishes for all time if and only if the corresponding component of the momentum of the center of mass is