11

Introduction to Advanced Dynamics

11.1. Introduction

At the age of 19, recognized for his extraordinary mathematical abilities, Joseph Louis de Lagrange (1736–1813) was appointed professor of geometry and mechanics at the Royal Artillery School at Turin, Italy, his birthplace. Here he developed his method of variations, invented earlier by Euler (in 1744) who later named it the calculus of variations. Lagrange left Turin in 1766 to become director of the Berlin Academy of Sciences until 1787 when, at the invitation of King Louis XVI of France, he was appointed to the Paris Academy of Sciences. Shortly thereafter his most celebrated work, *Mécanique Analytique*, appeared in 1788, nearly a century after the appearance of Newton’s *Principia*. Therein, Lagrange sets down an energy based approach for dynamics—the analysis of motion.

Inspired and strongly influenced by his senior contemporaries D’Alembert (1717–1785) and Euler (1707–1783), Lagrange linked the classical concepts and postulates of others in an invariant formulation of the equations of classical mechanics, now known as Lagrange’s equations. The method begins with construction of a single scalar function of the total kinetic and potential energies, called the Lagrangian function, and for general dynamical systems it employs the method of virtual work to identify the nonconservative generalized forces. Although Lagrange’s analytical mechanics embraces the theories of Newton and Euler, as it must, but in terms of work and energy, we shall see that it does not explicitly identify specific concepts of momentum, moment of momentum, center of mass, and rigidity. With these classical concepts in hand, Lagrange’s method provides a systematic scheme for the formulation of the equations of motion and

†† Lagrange’s life and times are sketched in the translators’ “Introduction” in Lagrange’s *Analytical Mechanics* cited in the chapter References. See also Truesdell’s *Essays*. 

495
their first integrals for any multidegree of freedom dynamical system consisting of any number of particles and rigid bodies.

Although a detailed study of Lagrange’s analytical dynamics is beyond the scope of this Introduction, still, we can accomplish a great deal. Our objective is to derive Lagrange’s equations of motion for all sorts of classical (holonomic) dynamical systems, both conservative and nonconservative, consisting of a particle, a system of particles, a rigid body, several connected rigid bodies, in fact, any combination of these objects. First, various kinds of system constraints are discussed. Then, Lagrange’s equations of motion for a particle are formulated and illustrated in some applications. Their straightforward extension for a system of particles follows. Hamilton’s principle of stationary action, a method based upon the calculus of variations, is introduced, and Lagrange’s equations are then derived from this principle without mention of any specific dynamical system. A number of examples are exhibited along the way.

11.2. Generalized Coordinates, Degrees of Freedom, and Constraints

We begin with a description of degrees of freedom and system constraints. Recall from Chapter 2 that the number of degrees of freedom of a dynamical system is the number of independent coordinates required to uniquely specify the location and orientation of all material points of the system relative to an assigned reference frame. A rigid disk free to move in the xy-plane, for example, has three degrees of freedom; two coordinates \((x_p, y_p)\) specify the location of any disk point \(P\) and one coordinate \(\theta\) provides the angle of rotation of the disk about its normal axis, say. If \(P\) is constrained to move on a specified path \(y_p = f(x_p)\), only two coordinates \(x_p\) and \(\theta\) are independent and hence the disk now has two degrees of freedom. In general, if there are \(c\) independent kinematical constraint equations relating the \(n\) coordinates, there remain \(n - c = d\) independent coordinates, i.e. degrees of freedom.

The number of degrees of freedom is strictly a property of the system; it is independent of the particular coordinates used to uniquely specify the configuration of the system. Imagine, for example, that the dynamical system requires \(m\) Cartesian coordinates \(x_k, k = 1, 2, \ldots, m\), to uniquely specify its configuration in a Cartesian frame \(\psi\) at an instant \(t\), and these \(m\) coordinates are related by \(r\) kinematical equations of constraint. Then \(d = m - r\). Suppose, on the other hand, that the \(x_k\) coordinates are related to another set of \(p\) generalized coordinates \(q_l, l = 1, 2, \ldots, p\), that uniquely specify the system configuration in \(\psi\) at the instant \(t\) so that, in general, \(x_k = x_k(q_1, q_2, \ldots, q_p, t) \equiv x_k(q_l, t)\), say. These equations describe the transformation from the set of ordinary coordinates \(x_k\) to the set of generalized coordinates \(q_l\) for a fixed \(t\). If the new coordinates \(q_k\) are related by \(s\) kinematical equations of constraint, then, regardless of the particular set of coordinates used to specify the configuration in \(\psi\) at time \(t, d = m - r = p - s\), the number of degrees of freedom of the system is the same. It may not be possible,